

# A REPRODUCING KERNEL THESIS FOR OPERATORS ON $\ell^2$ -VALUED BERGMAN-TYPE FUNCTION SPACES

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**ABSTRACT.** In this paper we consider the reproducing kernel thesis for boundedness and compactness for operators on  $\ell^2$ -valued Bergman-type spaces. This paper generalizes many well-known results about classical function spaces to their  $\ell^2$ -valued versions. In particular, the results in this paper apply to the weighted  $\ell^2$ -valued Bergman space on the unit ball, the unit polydisc and, more generally to weighted Fock spaces.

## 1. INTRODUCTION

In [21], Mitkovski and Wick show that in a wide variety of classical functions spaces (they call these spaces Bergman-type function spaces), many properties of an operator can be determined by studying its behavior on the normalized reproducing kernels. Thus, their results are “Reproducing Kernel Thesis” (RKT) statements.

The unified approach developed in [21] was used to solve two types of problems relating to operators on classical function spaces: boundedness and compactness. The goal of this paper is to extend this approach to the case of  $\ell^2$ -valued Bergman type function spaces and to prove results relating to boundedness and compactness of operators for a general class of  $\ell^2$ -valued Bergman type function spaces.

The proofs in this paper are essentially the same as the corresponding proofs from [21]. The only adjustments are that our integrals are now vector-valued and we must use a version of the classical Schur’s test for integral operators with matrix-valued kernels. This is Lemma 2.5. While this lemma is not deep, and is probably known (or at least expected) by experts, we were unable to find it in the literature.

The paper is organized as follows. In Section 2, we give a precise definition of  $\ell^2$ -valued Bergman-type spaces and prove some of their basic properties. In Section 3, we prove RKT statements for boundedness and extend several classical results about Toeplitz and Hankel operators to the  $\ell^2$ -valued setting. In Section 4, we prove RKT statements for compactness. In the final section, Section 5, we show that an operator is compact if and only if it is in the Toeplitz algebra and its Berezin transform vanishes on the boundary of  $\Omega$ .

## 2. $\ell^2$ -VALUED BERGMAN-TYPE SPACES

Before we can define the  $\ell^2$ -valued Bergman-type spaces, we will need to make some general definitions regarding  $\ell^2$ -valued functions. Let  $\Omega$  be a domain (connected open set) in  $\mathbb{C}^n$ , let  $\mu$  be a measure on  $\Omega$  and let  $\{e_k\}_{k=1}^\infty$  be the standard orthonormal basis for  $\ell^2$ . We say that function  $f : \Omega \rightarrow \ell^2$  is  $\mu$ -measurable (analytic) if for each  $k \in \mathbb{N}$  the function

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2000 *Mathematics Subject Classification.* 32A36, 32A, 47B05, 47B35.

*Key words and phrases.* Berezin Transform, Compact Operators, Bergman Space, Essential Norm, Toeplitz Algebra, Toeplitz Operator, vector-valued Bergman Space.

$z \mapsto \langle f(z), e_k \rangle_{\ell^2}$  is  $\mu$ -measurable (analytic) on  $\Omega$ . For a  $\mu$ -measurable set  $E \subset \Omega$ , and a  $\mu$ -measurable function  $f$ , we define the integral of  $f$  over  $E$ :

$$\int_E f d\mu := \sum_{k=1}^{\infty} \left( \int_E \langle f, e_k \rangle_{\ell^2} d\mu \right) e_k.$$

This is a well-defined element of  $\ell^2$  whenever

$$\sum_{k=1}^{\infty} \left| \int_E \langle f, e_k \rangle_{\ell^2} d\mu \right|^2 < \infty.$$

The space  $L^2(\Omega, \mathbb{C}; \mu)$  is the space of all  $\mathbb{C}$ -valued  $\mu$ -measurable functions,  $g$ , such that

$$\|g\|_{L^2(\Omega, \mathbb{C}; \mu)}^2 := \int_{\Omega} |g|^2 d\mu < \infty.$$

The space  $L^2(\Omega, \ell^2; \mu)$  is the space of all  $\ell^2$ -valued measurable functions,  $f$ , such that

$$\|f\|_{L^2(\Omega, \ell^2; \mu)}^2 := \int_{\Omega} \|f\|_{\ell^2}^2 d\mu.$$

Note that  $L^2(\Omega, \ell^2; \mu)$  is a Hilbert space with inner product:

$$\langle f, g \rangle_{L^2(\Omega, \ell^2; \mu)} := \int_{\Omega} \langle f, g \rangle_{\ell^2} d\mu$$

and in this case

$$\|f\|_{L^2(\Omega, \ell^2; \mu)}^2 = \sum_{k=1}^{\infty} \int_{\Omega} |\langle f, e_k \rangle_{\ell^2}|^2 d\mu = \sum_{k=1}^{\infty} \|\langle f, e_k \rangle\|_{L^2(\Omega, \mathbb{C}; d\mu)}^2,$$

where  $\langle \cdot, \cdot \rangle_{\ell^2}$  is the standard inner product on  $\ell^2$ . The spaces we will consider in this paper are spaces of functions that take values in  $\ell^2$ . However, we will also have occasion to discuss some spaces of  $\ell^p$ -valued functions. We will refer to such spaces as “vector-valued function spaces”.

The space  $L^p(\Omega, \mathbb{C}; \mu)$  is the space of all  $\mathbb{C}$ -valued  $\mu$ -measurable functions,  $g$ , such that

$$\|g\|_{L^p(\Omega, \mathbb{C}; \mu)}^p := \int_{\Omega} |g|^p d\mu < \infty.$$

The space  $L^p(\Omega, \ell^p; \mu)$  is the space of all  $\ell^p$ -valued measurable functions,  $f$ , such that

$$\|f\|_{L^p(\Omega, \ell^p; \mu)}^p := \int_{\Omega} \|f\|_{\ell^p}^p d\mu = \sum_{k=1}^{\infty} \int_{\Omega} |\langle f, e_k \rangle_{\ell^2}|^p d\mu.$$

The functions  $\|\cdot\|_{L^p(\Omega, \ell^p; \mu)}$  clearly satisfy  $\|\lambda f\|_{L^p(\Omega, \ell^p; \mu)} = \lambda \|f\|_{L^p(\Omega, \ell^p; \mu)}$  and the triangle inequality. If we identify two functions if  $\|f(z) - g(z)\|_{\ell^p} = 0$  for  $\mu$ -a.e.  $z \in \Omega$  then  $\|\cdot\|_{L^p(\Omega, \ell^p; \mu)}$  is positive definite. Therefore, the functions  $\|\cdot\|_{L^p(\Omega, \ell^p; \mu)}$  define norms. The spaces are also complete (see, for example, [12]) and so they are all Banach spaces.

We introduce a large class of  $\ell^2$ -valued reproducing kernel Hilbert spaces that will form an abstract framework for our results. Due to their similarities with the classical Bergman space we call them  $\ell^2$ -valued Bergman-type spaces. In defining the key properties of these spaces, we use the standard notation that  $A \lesssim B$  to denote that there exists a constant  $C$  such that  $A \leq CB$ . And,  $A \simeq B$  which means that  $A \lesssim B$  and  $B \lesssim A$ .

Below we list the defining properties of these spaces.

- A.1 Let  $\Omega$  be a domain (connected open set) in  $\mathbb{C}^n$  which contains the origin. We assume that for each  $z \in \Omega$ , there exists an involution  $\varphi_z \in \text{Aut}(\Omega)$  satisfying  $\varphi_z(0) = z$ .
- A.2 We assume the existence of a metric  $\mathfrak{d}$  on  $\Omega$  which is quasi-invariant under  $\varphi_z$ , i.e.,  $\mathfrak{d}(u, v) \simeq \mathfrak{d}(\varphi_z(u), \varphi_z(v))$  with the implied constants independent of  $u, v \in \Omega$ . In addition, we assume that the metric space  $(\Omega, \mathfrak{d})$  is separable and finitely compact, i.e., every closed ball in  $(\Omega, \mathfrak{d})$  is compact. As usual, we denote by  $D(z, r)$  the disc centered at  $z$  with radius  $r$  with respect to the metric  $\mathfrak{d}$ .
- A.3 We assume the existence of a finite Borel measure  $\sigma$  on  $\Omega$  and define  $\mathcal{B}(\Omega)$  to be the space of  $\ell^2$ -valued analytic functions on  $\Omega$  equipped with the  $L^2(\Omega, \ell^2; d\sigma)$  norm. We shall also have occasion to consider the space of  $\mathbb{C}$ -valued analytic functions on  $\Omega$  that are also in  $L^2(\Omega, \mathbb{C}; d\sigma)$ . We will denote this space by  $\mathcal{B}(\Omega, \mathbb{C})$ . Note that this space is the “scalar-valued” Bergman-type space as defined in [21]. Everywhere in the paper,  $\|\cdot\|_{\mathcal{B}(\Omega)}$  and  $\langle \cdot, \cdot \rangle_{\mathcal{B}(\Omega)}$  will denote the norm and the inner product in  $L^2(\Omega, \ell^2; d\sigma)$  and  $\|\cdot\|_{\mathcal{B}(\Omega, \mathbb{C})}$  and  $\langle \cdot, \cdot \rangle_{\mathcal{B}(\Omega, \mathbb{C})}$  will always denote the norm and the inner product in  $L^2(\Omega, \mathbb{C}; d\sigma)$ . We assume that  $\mathcal{B}(\Omega)$  is a reproducing kernel Hilbert space (RKHS) and denote by  $K_z$  and  $k_z$  the reproducing and the normalized reproducing kernels in  $\mathcal{B}(\Omega, \mathbb{C})$ . That is, for every  $g \in \mathcal{B}(\Omega, \mathbb{C})$ , and every  $z \in \Omega$  there holds:

$$g(z) = \int_{\Omega} \overline{K_z(w)} g(w) d\sigma(w).$$

And for every  $f \in \mathcal{B}(\Omega)$  and  $z \in \Omega$ , there holds:

$$f(z) = \int_{\Omega} \overline{K_z(w)} f(w) d\sigma(w) = \int_{\Omega} \langle K_w, K_z \rangle_{\mathcal{B}(\Omega, \mathbb{C})} f(w) d\sigma(w).$$

To emphasize, the reproducing kernels  $K_z$  are  $\mathbb{C}$ -valued analytic functions in  $\mathcal{B}(\Omega, \mathbb{C})$  and they act as reproducing kernels on both spaces  $\mathcal{B}(\Omega)$  and  $\mathcal{B}(\Omega, \mathbb{C})$ . We will also assume that  $\|K_z\|_{\mathcal{B}(\Omega, \mathbb{C})}$  is continuous as a function of  $z$  taking  $(\Omega, \mathfrak{d})$  into  $\mathbb{R}$ .

We will say that  $\mathcal{B}(\Omega)$  is an  $\ell^2$ -valued Bergman-type space if in addition to A.1-A.3 it also satisfies the following properties.

- A.4 We assume that the measure  $d\lambda(z) := \|K_z\|_{\mathcal{B}(\Omega, \mathbb{C})}^2 d\sigma(z)$  is quasi-invariant under all  $\varphi_z$ , i.e., for every Borel set  $E \subset \Omega$  we have  $\lambda(E) \simeq \lambda(\varphi_z(E))$  with the implied constants independent of  $z \in \Omega$ . In addition, we assume that  $\lambda$  is doubling, i.e., there exists a constant  $C > 1$  such that for all  $z \in \Omega$  and  $r > 0$  we have  $\lambda(D(z, 2r)) \leq C\lambda(D(z, r))$ .
- A.5 We assume that

$$\left| \langle k_z, k_w \rangle_{\mathcal{B}(\Omega, \mathbb{C})} \right| \simeq \frac{1}{\|K_{\varphi_z(w)}\|_{\mathcal{B}(\Omega, \mathbb{C})}},$$

with the implied constants independent of  $z, w \in \Omega$ .

- A.6 We assume that there exists a positive constant  $\kappa < 2$  such that

$$\int_{\Omega} \frac{\left| \langle K_z, K_w \rangle_{\mathcal{B}(\Omega, \mathbb{C})} \right|^{\frac{r+s}{2}}}{\|K_z\|_{\mathcal{B}(\Omega, \mathbb{C})}^s \|K_w\|_{\mathcal{B}(\Omega, \mathbb{C})}^r} d\lambda(w) \leq C = C(r, s) < \infty, \quad \forall z \in \Omega \quad (2.1)$$

for all  $r > \kappa > s > 0$  or that (2.1) holds for all  $r = s > 0$ . In the latter case we will say that  $\kappa = 0$ . These will be called the Rudin-Forelli estimates for  $\mathcal{B}(\Omega, \mathbb{C})$ .

- A.7 We assume that  $\lim_{\mathfrak{d}(z, 0) \rightarrow \infty} \|K_z\|_{\mathcal{B}(\Omega, \mathbb{C})} = \infty$ .

We say that  $\mathcal{B}(\Omega)$  is a *strong  $\ell^2$ -valued Bergman-type space* if we have  $=$  instead of  $\simeq$  everywhere in A.1-A.5.

**2.1. Some Examples.** The classical Bergman spaces on the unit ball, polydisc, or over any bounded symmetric domain that satisfies the Rudin–Forelli estimates are all examples of scalar-valued Bergman-type spaces. It should be pointed out that in classical Bergman spaces on the ball, the invariant measure of A.4 is not strictly doubling. However, the only place where the doubling property is used is in geometric decomposition of  $\Omega$  in Proposition 2.9. However, results of this type are well known for the classical Bergman spaces on the ball. See for example [4, 8, 21, 22, 31].

Additionally, the classical Fock space is a scalar-valued Bergman-type space. For a more detailed discussion of examples of Bergman-type spaces, see [21]. Clearly, any Bergman-type space can be extended to a  $\ell^2$ -valued Bergman space and so the  $\ell^2$ -valued versions of these spaces are  $\ell^2$ -valued Bergman-type spaces.

**2.2. Classical Results Extended to the  $\ell^2$ -Valued Setting.** Before going on, we discuss notation. If  $\mathcal{X}$  and  $\mathcal{Y}$  are Banach spaces,  $\mathcal{L}(\mathcal{X}, \mathcal{Y})$  is the space of bounded linear operators from  $\mathcal{X}$  to  $\mathcal{Y}$  equipped with the usual operator norm. When  $\mathcal{X} = \mathcal{Y}$  we will write  $\mathcal{L}(\mathcal{X}, \mathcal{Y}) = \mathcal{L}(\mathcal{X})$ .

The symbols  $\|\cdot\|$  and  $\langle \cdot, \cdot \rangle$  will be used in several different ways throughout the paper. To make things clear, we will adorn these symbols with a subscript to indicate the space in which the norm or inner product is being taken.

For the rest of the paper, let  $\{e_k\}_{k=1}^\infty$  denote the standard orthonormal basis for  $\ell^2$ . If  $v$  is an element of  $\ell^2$ , then  $v_k$  will denote the  $k^{\text{th}}$  component of  $v$ . That is  $v_k = \langle v, e_k \rangle_{\ell^2}$ . Similarly, if  $f$  is an  $\ell^2$ -valued function,  $f_k$  will denote the  $k^{\text{th}}$  component function. That is,  $f_k(z) = \langle f(z), e_k \rangle_{\ell^2}$ .

The identity operator on  $\ell^2$  will be denoted by  $I$ . In addition, if  $d \in \mathbb{N}$ ,  $I^{(d)}$  is the operator that is the orthogonal projection onto the span of  $\{e_1, \dots, e_d\}$ . That is,  $I^{(d)}$  is the identity matrix with the first  $d$  entries on the diagonal set equal to 1 and all other entries set to 0. Also,  $I_{(d)}$  will be the “opposite” of  $I^{(d)}$ . That is,  $I_{(d)} = I - I^{(d)}$ .

If  $e \in \ell^2$ , we say that  $e$  is  $d$ -finite if there is a  $d \in \mathbb{N}$  such that  $e = I^{(d)}e$ . That is, only the first  $d$  entries of  $e$  may be non-zero. An operator  $U \in \mathcal{L}(\ell^2)$  will be called  $d$ -finite if there is a  $d \in \mathbb{N}$  such that  $U = I^{(d)}UI^{(d)}$ . Equivalently,  $\langle Ue_i, e_k \rangle_{\ell^2} = 0$  if either  $i > d$  or  $k > d$ . An  $\ell^2$ -valued function  $f : \Omega \rightarrow \ell^2$  will be called  $d$ -finite if  $f(z)$  is  $d$ -finite for all  $z \in \Omega$ . A matrix-valued function  $U : \Omega \rightarrow \mathcal{L}(\ell^2)$  will be called  $d$ -finite if  $U(z)$  is a  $d$ -finite operator on  $\ell^2$  for every  $z \in \Omega$ . If the exact value of  $d$  is not important, we will simply say “finite” instead of  $d$ -finite. For example, a vector  $u \in \ell^2$  is finite if there is a  $d$  such that  $u$  is  $d$ -finite.

We will often refer to linear operators on  $\ell^2$  (not necessarily bounded) as matrices. There should be no confusion that these matrices are infinite dimensional matrices and are written relative to the standard orthonormal basis  $\{e_k\}_{k=1}^\infty$ .

Let  $\mathcal{C} = \{f_a\}_{a \in \mathcal{A}}$  be a collection of  $\mathbb{C}$ -valued functions. A linear combination of functions in the collection  $\{f_a\}_{a \in \mathcal{A}}$  is a sum of the form:

$$f_1 h_1 + \dots + f_m h_m, \tag{2.2}$$

where each  $f_i \in \mathcal{C}$ , each  $h_i$  is a finite element of  $\ell^2$  and  $m < \infty$ . A  $\mathbb{C}$ -linear combination of functions in the collection is a sum of the form:

$$f_1 c_1 + \dots + f_m c_m,$$

where the  $c_i$  are complex numbers. To reiterate, whenever we say “linear combination”, we will mean one as defined in (2.2) so that a linear combination of scalar-valued functions is an  $\ell^2$ -valued function.

**Lemma 2.1.** *Let  $\mathcal{C}$  be a collection of  $\mathbb{C}$ -valued functions such that the set of  $\mathbb{C}$ -linear combinations of functions in  $\mathcal{C}$  is dense in  $\mathcal{B}(\Omega, \mathbb{C})$ . Then the linear combinations of elements of  $\mathcal{C}$  is dense in  $\mathcal{B}(\Omega)$ .*

*Proof.* First, let  $g \in \mathcal{B}(\Omega)$  be finite. Then since the  $\mathbb{C}$ -linear combinations of elements of  $\mathcal{C}$  are dense in  $\mathcal{B}(\Omega, \mathbb{C})$ , we can approximate  $g$  in the  $\mathcal{B}(\Omega)$  norm with linear combinations of elements of  $\mathcal{C}$ . Let  $f \in \mathcal{B}(\Omega)$  be arbitrary. Then  $\|f\|_{\mathcal{B}(\Omega)}^2 = \sum_k \|\langle f, e_k \rangle_{\ell^2}\|_{\mathcal{B}(\Omega, \mathbb{C})}^2 < \infty$ . Thus there is an  $N \in \mathbb{N}$  such that  $\sum_{k=N}^{\infty} \|\langle f, e_k \rangle_{\ell^2}\|_{\mathcal{B}(\Omega, \mathbb{C})}^2 < \epsilon$  and so  $\left\|f - \sum_{k=1}^{N-1} \langle f, e_k \rangle_{\ell^2}\right\|_{\mathcal{B}(\Omega)} < \epsilon$ . That is,  $f$  can be approximated by finite elements of  $\mathcal{B}(\Omega)$  in the  $\mathcal{B}(\Omega)$  norm. Let  $g$  be a finite element of  $\mathcal{B}(\Omega)$  such that  $\|f - g\|_{\mathcal{B}(\Omega)} \leq \epsilon$  and let  $h$  be a linear combination of elements of  $\mathcal{C}$  such that  $\|g - h\|_{\mathcal{B}(\Omega)} \leq \epsilon$ . Then there holds:

$$\|f - h\|_{\mathcal{B}(\Omega)} \leq \|f - g\|_{\mathcal{B}(\Omega)} + \|g - h\|_{\mathcal{B}(\Omega)} \leq 2\epsilon.$$

This completes the proof.  $\square$

This implies the following corollary:

**Corollary 2.2.** *The linear combinations of the normalized reproducing kernels, reproducing kernels, and monomials are all dense in  $\mathcal{B}(\Omega)$ .*

**2.3. Projection Operators on Bergman-type Spaces.** It is easy to see that the orthogonal projection of  $L^2(\Omega, \ell^2; d\sigma)$  onto  $\mathcal{B}(\Omega)$  is given by the integral operator

$$P(f)(z) := \int_{\Omega} \langle K_w, K_z \rangle_{\mathcal{B}(\Omega, \mathbb{C})} f(w) d\sigma(w).$$

Therefore, for all  $f \in \mathcal{B}(\Omega)$  we have  $f(z) = \int_{\Omega} k_w(z) \langle f, k_w \rangle_{\ell^2} d\lambda(w)$ . Moreover,

$$\|f\|_{\mathcal{B}(\Omega)}^2 = \int_{\Omega} \langle f(w), f(w) \rangle_{\ell^2} d\sigma(w) = \int_{\Omega} \sum_{j=1}^{\infty} |\langle f_j, k_w \rangle_{\ell^2}|^2 d\lambda(w).$$

If  $\kappa > 0$ ,  $P$  is bounded as an operator on  $L^p(\Omega, \ell^p; d\sigma)$  for  $1 < p < \infty$  and if  $\kappa = 0$ ,  $P$  is bounded as an operator on  $L^p\left(\Omega, \ell^p; \frac{d\sigma(w)}{\|K_w\|_{\mathcal{B}(\Omega, \mathbb{C})}^p}\right)$ . In [21], the authors prove:

**Lemma 2.3.** *Let  $P(f)(z) := \int_{\Omega} \langle K_w, K_z \rangle_{\mathcal{B}(\Omega, \mathbb{C})} f(w) d\sigma(w)$  be the projection operator on  $L^p(\Omega, \mathbb{C}; d\sigma)$ .*

- (a) *If  $\kappa = 0$  then  $P$  is bounded as an operator from  $L^p\left(\Omega, \mathbb{C}; \frac{d\lambda(w)}{\|K_w\|_{\mathcal{B}(\Omega, \mathbb{C})}^p}\right)$  into  $L^p\left(\Omega, \mathbb{C}; \frac{d\lambda(w)}{\|K_w\|_{\mathcal{B}(\Omega, \mathbb{C})}^p}\right)$  for all  $1 \leq p \leq \infty$ .*
- (b) *If  $\kappa > 0$  then  $P$  is bounded as an operator from  $L^p(\Omega, \mathbb{C}; d\sigma)$  into  $L^p(\Omega, \mathbb{C}; d\sigma)$  for all  $1 < p < \infty$ .*

The proof of the following lemma is easily deduced from Lemma 2.3 and is omitted.

**Lemma 2.4.** *Let  $P(f)(z) := \int_{\Omega} \langle K_w, K_z \rangle_{\mathcal{B}(\Omega, \mathbb{C})} f(w) d\sigma(w)$  be the projection operator.*

- (a) *If  $\kappa = 0$  then  $P$  is bounded as an operator from  $L^p\left(\Omega, \ell^p; \frac{d\lambda(w)}{\|K_w\|_{\mathcal{B}(\Omega, \mathbb{C})}^p}\right)$  into  $L^p\left(\Omega, \ell^p; \frac{d\lambda(w)}{\|K_w\|_{\mathcal{B}(\Omega, \mathbb{C})}^p}\right)$  for all  $1 \leq p \leq \infty$ .*

(b) If  $\kappa > 0$  then  $P$  is bounded as an operator from  $L^p(\Omega, \ell^p; d\sigma)$  into  $L^p(\Omega, \ell^p; d\sigma)$  for all  $1 < p < \infty$

The following is a matrix version of the classical Schur's Test.

**Lemma 2.5** (Schur's Test for Matrix-Valued Kernels). *Let  $(X, \mu)$  and  $(X, \nu)$  be measure spaces and  $M(x, y)$  a measurable matrix-valued function on  $X \times X$  whose entries are non-negative. That is, for all  $k, i \in \mathbb{N}$  there holds:*

$$\langle M(x, y)e_k, e_i \rangle_{\ell^2} \geq 0.$$

*If  $h$  is a positive measurable function (with respect to  $\mu$  and  $\nu$ ), and if  $C_1, C_2$  are positive constants such that*

$$\begin{aligned} \int_X \sum_{k=1}^{\infty} h(y)^q \langle M(x, y)e_k, e_i \rangle_{\ell^2} d\nu(y) &\leq C_1 h(x)^q \text{ for } \mu\text{-almost every } x; \\ \int_X \sum_{i=1}^{\infty} h(x)^p \langle M^*(x, y)e_i, e_k \rangle_{\ell^2} d\mu(x) &\leq C_2 h(y)^p \text{ for } \nu\text{-almost every } y, \end{aligned}$$

*then  $Tf(x) = \int_X M(x, y)f(y)d\nu(y)$  defines a bounded operator  $T : L^p(X, \ell^p; \nu) \rightarrow L^p(X, \ell^p; \mu)$  with norm no greater than  $C_1^{1/q}C_2^{1/p}$ .*

*Proof.* The proof is simply an appropriate adaptation of a standard proof for the classical Schur's Test. The following computation uses Hölder's Inequality at the level of the integral and at the level of the infinite sum, we also use the first assumption:

$$\begin{aligned} |(Tf_i)(x)| &= |\langle Tf(x), e_i \rangle_{\ell^2}| \\ &\leq \int_X \sum_{k=1}^{\infty} h(y)h(y)^{-1} |f_k(y)| \langle M(x, y)e_k, e_i \rangle_{\ell^2} d\nu(y) \\ &\leq \int_X \left\{ \sum_{k=1}^{\infty} h^q(y) \langle M(x, y)e_k, e_i \rangle_{\ell^2} \right\}^{\frac{1}{q}} \left\{ \sum_{k=1}^{\infty} h^{-p}(y) |f_k(y)|^p \langle M(x, y)e_k, e_i \rangle_{\ell^2} \right\}^{\frac{1}{p}} d\nu(y) \\ &\leq \left\{ \int_X \sum_{k=1}^{\infty} h(y)^q \langle M(x, y)e_k, e_i \rangle_{\ell^2} d\nu(y) \right\}^{\frac{1}{q}} \left\{ \int_X \sum_{k=1}^{\infty} h^{-p}(y) |f_k(y)|^p \langle M(x, y)e_k, e_i \rangle_{\ell^2} d\nu(y) \right\}^{\frac{1}{p}} \\ &\leq C_1^{\frac{1}{q}} h(x) \left\{ \sum_{k=1}^{\infty} \int_X h^{-p}(y) |f_k(y)|^p \langle M(x, y)e_k, e_i \rangle_{\ell^2} d\nu(y) \right\}^{\frac{1}{p}}. \end{aligned}$$

Using the above estimate and the second assumption, there holds:

$$\begin{aligned} \|Tf\|_{L^p(X, \ell^p; \mu)}^p &= \int \sum_{i=1}^{\infty} |\langle Tf(x), e_i \rangle_{\ell^2}|^p d\mu(x) \\ &\leq \int_X \sum_{i=1}^{\infty} \left\{ C_1^{\frac{1}{q}} h(x) \left( \int_X \sum_{k=1}^{\infty} h^{-p}(y) |f_k(y)|^p \langle M(x, y)e_k, e_i \rangle_{\ell^2} d\nu(y) \right)^{\frac{1}{p}} \right\}^p d\mu(x) \\ &= C_1^{\frac{p}{q}} \int_X \sum_{k=1}^{\infty} |f_k(y)|^p h^{-p}(y) \int_X \sum_{i=1}^{\infty} h^p(x) \langle M^*(x, y)e_i, e_k \rangle_{\ell^2} d\mu(x) d\nu(y) \end{aligned}$$

$$\begin{aligned} &\leq C_1^{\frac{p}{q}} C_2 \int_X \sum_{k=1}^{\infty} |f_k(y)|^p d\nu(y) \\ &= C_1^{\frac{p}{q}} C_2 \|f\|_{L^p(X, \ell^p; \nu)}^p. \end{aligned}$$

Now take  $p^{th}$  roots. The interchange of integrals and sums and the switching the order of integration are justified since the integrand is non-negative.  $\square$

The following result will be useful later when applying the Matrix Schur's Test, Lemma 2.5. See [21] for the proof.

**Lemma 2.6.** *For all  $r, s \in \mathbb{R}$  the following quasi-identity holds*

$$\int_{\Omega} \frac{|\langle K_z, K_w \rangle_{\mathcal{B}(\Omega, \mathbb{C})}|^{\frac{r-s}{2}}}{\|K_w\|_{\mathcal{B}(\Omega, \mathbb{C})}^r} d\lambda(w) \simeq \int_{\Omega} \frac{|\langle K_z, K_w \rangle_{\mathcal{B}(\Omega, \mathbb{C})}|^{\frac{r+s}{2}}}{\|K_z\|_{\mathcal{B}(\Omega, \mathbb{C})}^s \|K_w\|_{\mathcal{B}(\Omega, \mathbb{C})}^r} d\lambda(w) \quad (2.3)$$

where the implied constants are independent of  $z \in \Omega$  and may depend on  $r, s$ .

**2.4. Translation Operators on Bergman-type Spaces.** For each  $z \in \Omega$  we define an adapted translation operator  $U_z$  on  $\mathcal{B}(\Omega)$  by

$$U_z f(w) := f(\varphi_z(w)) k_z(w) = \sum_{k=1}^{\infty} \langle f \circ \varphi_z(w) k_z(w), e_k \rangle_{\ell^2} e_k.$$

Each  $U_z$  is invertible with the inverse given by

$$U_z^{-1} f(w) := \frac{1}{k_z(\varphi_z(w))} f(\varphi_z(w)) = \sum_{k=1}^{\infty} \left\langle \frac{1}{k_z(\varphi_z(w))} f \circ \varphi_z(w), e_k \right\rangle_{\ell^2} e_k.$$

The inverse also satisfies  $\|U_z^{-1} f\|_{\mathcal{B}(\Omega)} \simeq \|f\|_{\mathcal{B}(\Omega)}$ . Therefore, for every  $f \in \mathcal{B}(\Omega)$  there holds

$$\|f\|_{\mathcal{B}(\Omega)}^2 = \langle U_z^* f, U_z^{-1} f \rangle_{\mathcal{B}(\Omega)} \leq \|U_z^* f\|_{\mathcal{B}(\Omega)} \|U_z^{-1} f\|_{\mathcal{B}(\Omega)} \lesssim \|U_z^* f\|_{\mathcal{B}(\Omega)} \|f\|_{\mathcal{B}(\Omega)}.$$

This implies that also  $\|U_z^* f\|_{\mathcal{B}(\Omega)} \simeq \|f\|_{\mathcal{B}(\Omega)}$ . We will also use the symbols  $U_z$  to denote the operators on  $\mathcal{B}(\Omega, \mathbb{C})$  given by the formula:

$$U_z h(w) = h(\varphi_z(w)) k_z(w)$$

for every  $h \in \mathcal{B}(\Omega, \mathbb{C})$ . It will be clear from context which is meant.

**Lemma 2.7.** *The following quasi-equalities hold for all  $f \in \mathcal{B}(\Omega)$  and for all  $g \in \mathcal{B}(\Omega, \mathbb{C})$ :*

- (a)  $|U_z g| \simeq |g|$ ,
- (b)  $|U_z^2 g| \simeq |g|$ ,
- (c)  $|U_z^* k_w| \simeq |k_{\varphi_z(w)}|$ .
- (a')  $\|U_z f\|_{\mathcal{B}(\Omega)} \simeq \|f\|_{\mathcal{B}(\Omega)}$ ,
- (b')  $\|U_z^2 f\|_{\mathcal{B}(\Omega)} \simeq \|f\|_{\mathcal{B}(\Omega)}$ ,

*Proof.* Assertions (a)-(c) were proven in [20, Lemma 2.9]. We use them to prove assertions (a') and (b'). Note that

$$\|f\|_{\mathcal{B}(\Omega)}^2 = \sum_k \|\langle f, e_k \rangle_{\ell^2}\|_{\mathcal{B}(\Omega, \mathbb{C})}^2$$



and

$$U_z f(w) = \sum_k \langle f \circ \varphi_z(w) k_z(w), e_k \rangle_{\ell^2} e_k.$$

To prove (a'), there holds:

$$\begin{aligned} \|U_z f\|_{\mathcal{B}(\Omega)}^2 &= \left\| \sum_{k=1}^{\infty} \langle f \circ \varphi_z(w) k_z(w), e_k \rangle_{\ell^2} e_k \right\|_{\mathcal{B}(\Omega)}^2 = \sum_{k=1}^{\infty} \|\langle f \circ \varphi_z(w) k_z(w), e_k \rangle_{\ell^2}\|_{\mathcal{B}(\Omega, \mathbb{C})}^2 \\ &\simeq \sum_{k=1}^{\infty} \|\langle f, e_k \rangle_{\ell^2}\|_{\mathcal{B}(\Omega, \mathbb{C})}^2 \\ &= \|f\|_{\mathcal{B}(\Omega)}^2. \end{aligned}$$

Assertion (b') is proven similarly.  $\square$

In case of a strong  $\ell^2$ -valued Bergman-type space, the  $U_z$  are actually unitary operators. Moreover, in this case,  $U_z^2 = I$  and for  $u, w, z \in \Omega$  and  $e \in \ell^2$ , there holds

$$U_z(k_w e)(u) = U_z^*(k_w e)(u) = \langle k_w, k_w \rangle_{\mathcal{B}(\Omega, \mathbb{C})} \|K_{\varphi_z(w)}\|_{\mathcal{B}(\Omega, \mathbb{C})} k_{\varphi_z(w)}(u) e. \quad (2.4)$$

Since  $|\langle k_w, k_w \rangle_{\mathcal{B}(\Omega, \mathbb{C})}| \|K_{\varphi_z(w)}\|_{\mathcal{B}(\Omega, \mathbb{C})} = 1$ , this also implies that  $\|U_z^* k_w e\|_{\ell^2} = \|k_{\varphi_z(w)} e\|_{\ell^2} = \|U_z k_w e\|_{\ell^2}$ .

For any given operator  $T$  on  $\mathcal{B}(\Omega)$  and  $z \in \Omega$  we define  $T^z := U_z T U_z^*$ .

**2.5. Toeplitz Operators on  $\ell^2$ -Valued Bergman-type Spaces.** An operator-valued function  $u : \Omega \rightarrow \mathcal{L}(\ell^2)$  will be called measurable (analytic) if the function  $z \mapsto \langle u(z) e_k, e_i \rangle_{\ell^2}$  is measurable (analytic) for every  $i, k \in \mathbb{N}$ . Let  $u : \Omega \rightarrow \mathcal{L}(\ell^2)$  be measurable. Define  $M_u$  as the operator on  $\mathcal{B}(\Omega)$  given by the formula:

$$(M_u f)(z) = u(z) f(z).$$

Define the Toeplitz operator with symbol  $u$  by:

$$T_u := P M_u,$$

where  $P$  is the usual projection operator onto  $\mathcal{B}(\Omega)$ . Let  $L_{\mathcal{L}(\ell^2)}^\infty$  be the set of functions  $u : \Omega \rightarrow \mathcal{L}(\ell^2)$  such that  $w \mapsto \|u(w)\|_{\mathcal{L}(\ell^2)}$  is in  $L^\infty(\Omega, \mathbb{C}; d\sigma)$ . When  $u \in L_{\mathcal{L}(\ell^2)}^\infty$ , it is immediate to see that  $\|T_u\| \leq \|u\|_{L_{\mathcal{L}(\ell^2)}^\infty}$ . In the next section we will provide a condition on  $u$  which will guarantee that  $T_u$  is bounded.

We are going to further refine this class of Toeplitz operators. We say that the function  $u \rightarrow \mathcal{L}(\ell^2)$  is in  $L_{\text{fin}}^\infty$  if  $u$  is finite and  $u \in L_{\mathcal{L}(\ell^2)}^\infty$ . In other words, a function  $u \in L_{\text{fin}}^\infty$  may be viewed as a  $d \times d$  matrix-valued function with bounded entries. These Toeplitz operators are the key building blocks of an important object for this paper, the Toeplitz algebra, denoted by  $\mathcal{T}_{L_{\text{fin}}^\infty}$ , associated to the symbols in  $L_{\text{fin}}^\infty$ . Specifically, we define

$$\mathcal{T}_{L_{\text{fin}}^\infty} := \text{clos}_{\mathcal{L}(\mathcal{B}(\Omega))} \left\{ \sum_{l=1}^L \prod_{j=1}^J T_{u_{j,l}} : u_{j,l} \in L_{\text{fin}}^\infty, J, L \text{ finite} \right\}$$

where the closure is taken in the operator norm topology on  $\mathcal{L}(\mathcal{B}(\Omega))$ .

In the case of strong  $\ell^2$ -valued Bergman-type spaces, conjugation by translations behaves particularly well with respect to Toeplitz operators. Namely, if  $T = T_u$  is a Toeplitz operator



then  $T_u^z = T_{u \circ \varphi_z}$ . Moreover, when  $T = T_{u_1} T_{u_2} \cdots T_{u_n}$  is a product of Toeplitz operators there holds

$$T^z = T_{u_1 \circ \varphi_z} T_{u_2 \circ \varphi_z} \cdots T_{u_n \circ \varphi_z}.$$

The following lemma is easily deduced from [21, Lemma 2.10] and will be used in what follows.

**Lemma 2.8.** *For each bounded Borel set  $G$  in  $\Omega$ , and each  $d \in \mathbb{N}$ , the Toeplitz operator  $T_{1_G} M_{I^{(d)}} = M_{I^{(d)}} T_{1_G}$  is compact on  $\mathcal{B}(\Omega)$ .*

**2.6. Geometric Decomposition of  $(\Omega, \mathfrak{d}, \lambda)$ .** The proof of the crucial localization result from Section 4 will make critical use of the following covering result. For the proof see [21]. Related results can be found in [4, 8, 22, 31] where it is shown that nice domains, such as the unit ball, polydisc, or  $\mathbb{C}^n$  have this property.

**Proposition 2.9.** *There exists an integer  $N > 0$  (depending only on the doubling constant of the measure  $\lambda$ ) such that for any  $r > 0$  there is a covering  $\mathcal{F}_r = \{F_j\}$  of  $\Omega$  by disjoint Borel sets satisfying*

- (1) *every point of  $\Omega$  belongs to at most  $N$  of the sets  $G_j := \{z \in \Omega : \mathfrak{d}(z, F_j) \leq r\}$ ,*
- (2)  *$\text{diam}_{\mathfrak{d}} F_j \leq 4r$  for every  $j$ .*

### 3. REPRODUCING KERNEL THESES FOR BOUNDEDNESS

In this section, we will give sufficient conditions for boundedness of operators on  $\mathcal{B}(\Omega)$ . Ideally, we would like to show that the conditions:

$$\sup_k \sup_{z \in \Omega} \|U_z T k_z e_k\|_{L^p(\Omega, \ell^p; d\sigma)}^p = \sup_k \sup_{z \in \Omega} \sum_{i=1}^{\infty} \|\langle U_z T(k_z e_k), e_i \rangle\|_{L^p(\Omega, \mathbb{C}; d\sigma)}^p < \infty$$

and

$$\sup_i \sup_{z \in \Omega} \|U_z T^* k_z e_i\|_{L^p(\Omega, \ell^p; d\sigma)}^p = \sup_i \sup_{z \in \Omega} \sum_{k=1}^{\infty} \|\langle U_z T(k_z e_i), e_k \rangle\|_{L^p(\Omega, \mathbb{C}; d\sigma)}^p < \infty,$$

are enough to guarantee that  $T$  is bounded. However, if  $T$  satisfies a stronger condition, we can conclude that  $T$  is bounded.

**Theorem 3.1.** *Let  $T : \mathcal{B}(\Omega) \rightarrow \mathcal{B}(\Omega)$  be a linear operator defined a priori only on the linear span of normalized reproducing kernels of  $\mathcal{B}(\Omega)$ . Assume that there exists an operator  $T^*$  defined on the same span such that the duality relation  $\langle T k_z e, k_w h \rangle_{\mathcal{B}(\Omega)} = \langle k_z e, T^* k_w h \rangle_{\mathcal{B}(\Omega)}$  holds for all  $z, w \in \Omega$  and all finite  $e, h \in \ell^2$ . Let  $\kappa$  be the constant from A.6. If*

$$\sup_i \sup_{z \in \Omega} \left\{ \int_{\Omega} \left( \sum_{k=1}^{\infty} |\langle U_z T^*(k_z e_i)(u), e_k \rangle_{\ell^2}| \right)^p d\sigma(u) \right\}^{\frac{1}{p}} < \infty, \quad (3.1)$$

and

$$\sup_k \sup_{z \in \Omega} \left\{ \int_{\Omega} \left( \sum_{i=1}^{\infty} |\langle U_z T(k_z e_k)(u), e_i \rangle_{\ell^2}| \right)^p d\sigma(u) \right\}^{\frac{1}{p}} < \infty \quad (3.2)$$

for some  $p > \frac{4-\kappa}{2-\kappa}$  then  $T$  can be extended to a bounded operator on  $\mathcal{B}(\Omega)$ .

*Remark 3.2.* Note that by Minkowski's inequality, the above conditions can be replaced by

$$\sup_i \sup_{z \in \Omega} \sum_{k=1}^{\infty} \|\langle U_z T^*(k_z e_i), e_k \rangle_{\ell^2}\|_{L^p(\Omega, \mathbb{C}; d\sigma)} < \infty \quad (3.3)$$

and

$$\sup_k \sup_{z \in \Omega} \sum_{i=1}^{\infty} \|\langle U_z T(k_z e_k), e_i \rangle_{\ell^2}\|_{L^p(\Omega, \mathbb{C}; d\sigma)} < \infty. \quad (3.4)$$

We state the theorem with conditions (3.1) and (3.2) since they are, in general, smaller than the quantities in (3.3) and (3.4). Similar statements are true for all of the theorems in this section.

*Proof.* Since the linear span of the normalized reproducing kernels is dense in  $\mathcal{B}(\Omega)$  it will be enough to show that there exists a finite constant such that  $\|Tf\|_{\mathcal{B}(\Omega)} \lesssim \|f\|_{\mathcal{B}(\Omega)}$  for all  $f$  that are in the linear span of the normalized reproducing kernels. Notice first that for any such  $f$  there holds

$$\begin{aligned} \int_{\Omega} \|(Tf)(z)\|_{\ell^2}^2 d\sigma(z) &= \int_{\Omega} \left\| \sum_{i=1}^{\infty} \langle Tf, K_z e_i \rangle_{\mathcal{B}(\Omega)} e_i \right\|_{\ell^2}^2 d\sigma(z) \\ &= \int_{\Omega} \left\| \sum_{i=1}^{\infty} \int_{\Omega} \sum_{k=1}^{\infty} \langle f_k(w) e_k, T^* K_z e_i \rangle_{\ell^2} e_i d\sigma(w) \right\|_{\ell^2}^2 d\sigma(z) \\ &= \int_{\Omega} \left\| \sum_{i=1}^{\infty} \int_{\Omega} \sum_{k=1}^{\infty} f_k(w) \langle K_w e_k, T^* K_z e_i \rangle_{\mathcal{B}(\Omega)} e_i d\sigma(w) \right\|_{\ell^2}^2 d\sigma(z) \\ &\leq \int_{\Omega} \left\| \int_{\Omega} \sum_{i=1}^{\infty} \sum_{k=1}^{\infty} |f_k(w)| \left| \langle K_w e_k, T^* K_z e_i \rangle_{\mathcal{B}(\Omega)} \right| e_i d\sigma(w) \right\|_{\ell^2}^2 d\sigma(z) \quad (3.5) \\ &= \int_{\Omega} \left\| \int_{\Omega} M(z, w) |f(w)| d\sigma(w) \right\|_{\ell^2}^2 d\sigma(z). \end{aligned}$$

In (3.5), we use the fact that

$$\left\| \sum_{i=1}^{\infty} \lambda_i e_i \right\|_{\ell^2}^2 = \left\| \sum_{i=1}^{\infty} |\lambda_i| e_i \right\|_{\ell^2}^2$$

and we define

$$|f(w)| := \sum_{k=1}^{\infty} |\langle f_k(w), e_k \rangle_{\ell^2}| e_k.$$

Thus, we only need to show that the integral operator with matrix-valued kernel  $M(z, w)$  is bounded from  $L^2(\Omega, \ell^2; d\sigma) \rightarrow L^2(\Omega, \ell^2; d\sigma)$ , where

$$\langle M(z, w) e_k, e_i \rangle_{\ell^2} = \left| \langle K_w e_k, T^*(K_z e_i) \rangle_{\mathcal{B}(\Omega)} \right|.$$

The Matrix Schur's Test, (Lemma 2.5), will be used to prove that this operator is bounded. We set

$$\langle M(z, w) e_k, e_i \rangle_{\ell^2} = \left| \langle K_w e_k, T^*(K_z e_i) \rangle_{\mathcal{B}(\Omega)} \right|, \quad h(z) \equiv \|K_z\|_{\mathcal{B}(\Omega, \mathbb{C})}^{\alpha/2},$$

$$X = \Omega, \quad d\mu(z) = d\nu(z) = d\sigma(z).$$

If  $\kappa = 0$  set  $\alpha = \frac{4-2\kappa}{4-\kappa} = 1$ . If  $\kappa > 0$  choose  $\alpha \in (\frac{2}{p}, \frac{4-2\kappa}{4-\kappa})$  such that  $q(\alpha - \frac{2}{p}) < \kappa$ . The condition  $p > \frac{4-\kappa}{2-\kappa}$  ensures that such  $\alpha$  exists. Let  $z \in \Omega$  be arbitrary and fixed. There holds

$$\begin{aligned} Q_1 &:= \int_{\Omega} \sum_{k=1}^{\infty} \|K_w\|_{\mathcal{B}(\Omega, \mathbb{C})}^{\alpha} \langle M(z, w)e_k, e_i \rangle_{\ell^2} d\sigma(w) \\ &= \int_{\Omega} \sum_{k=1}^{\infty} \left| \langle K_w e_k, T^*(K_z e_i) \rangle_{\mathcal{B}(\Omega)} \right| \|K_w\|_{\mathcal{B}(\Omega, \mathbb{C})}^{\alpha} d\sigma(w) \\ &= \|K_z\|_{\mathcal{B}(\Omega, \mathbb{C})} \int_{\Omega} \sum_{k=1}^{\infty} \left| \langle T^*(k_z e_i), k_{\varphi_z(u)} \rangle_{\mathcal{B}(\Omega)} \right| \|K_{\varphi_z(u)}\|_{\mathcal{B}(\Omega, \mathbb{C})}^{\alpha-1} d\lambda(w) \\ &\simeq \|K_z\|_{\mathcal{B}(\Omega, \mathbb{C})} \int_{\Omega} \sum_{k=1}^{\infty} \left| \langle T^*(k_z e_i), U_z^* k_u e_k \rangle_{\mathcal{B}(\Omega)} \right| \left| \langle k_z, k_u \rangle_{\mathcal{B}(\Omega, \mathbb{C})} \right|^{1-\alpha} d\lambda(w) \\ &= \|K_z\|_{\mathcal{B}(\Omega, \mathbb{C})} \int_{\Omega} \sum_{k=1}^{\infty} \left| \langle U_z T^*(k_z e_i), k_u e_k \rangle_{\mathcal{B}(\Omega)} \right| \left| \langle k_z, k_u \rangle_{\mathcal{B}(\Omega, \mathbb{C})} \right|^{1-\alpha} d\lambda(w) \\ &= \|K_z\|_{\mathcal{B}(\Omega, \mathbb{C})}^{\alpha} \int_{\Omega} \sum_{k=1}^{\infty} |\langle U_z T^*(k_z e_i)(u), e_k \rangle_{\ell^2}| \frac{\left| \langle K_z, K_u \rangle_{\mathcal{B}(\Omega, \mathbb{C})} \right|^{2-\alpha}}{\|K_u\|_{\mathcal{B}(\Omega, \mathbb{C})}^{2-\alpha}} d\lambda(u). \end{aligned}$$

By Hölder's Inequality, this quantity is no worse than:

$$\|K_z\|_{\mathcal{B}(\Omega, \mathbb{C})}^{\alpha} \left\{ \int_{\Omega} \left( \sum_{k=1}^{\infty} |\langle U_z T^*(k_z e_i)(u), e_k \rangle_{\ell^2}| \right)^p d\sigma(u) \right\}^{\frac{1}{p}} \left\{ \int_{\Omega} \frac{\left| \langle K_z, K_u \rangle_{\mathcal{B}(\Omega, \mathbb{C})} \right|^{q(1-\alpha)}}{\|K_u\|_{\mathcal{B}(\Omega, \mathbb{C})}^{q(2-\alpha-\frac{2}{p})}} d\lambda(u) \right\}^{\frac{1}{q}}.$$

Let  $r = q\left(2 - \alpha - \frac{2}{p}\right)$  and  $s = r - 2q(1 - \alpha)$ . Then  $r = \frac{p(2-\alpha-\frac{2}{p})}{p-1} = 2 - \frac{\alpha p}{p-1} > \kappa$  and  $s = q(\alpha - \frac{2}{p}) < \kappa$  when  $\kappa > 0$  and  $s = r > \kappa$  if  $\kappa = 0$ . This means that both  $r$  and  $s$  satisfy all condition of [A.6](#). Thus, by Lemma [2.6](#), the second integral is bounded independent of  $z$ . Call this constant  $C$ . This gives that:

$$Q_1 \leq C \|K_z\|_{\mathcal{B}(\Omega, \mathbb{C})}^{\alpha} \sup_i \sup_{z \in \Omega} \left\{ \int_{\Omega} \left( \sum_{k=1}^{\infty} |\langle U_z T^*(k_z e_i)(u), e_k \rangle_{\ell^2}| \right)^p d\sigma(u) \right\}^{\frac{1}{p}}.$$

By interchanging the roles of  $T$  and  $T^*$  and  $i$  and  $k$ , we similarly obtain:

$$\begin{aligned} Q_2 &:= \int_{\Omega} \sum_{i=1}^{\infty} \|K_z\|_{\mathcal{B}(\Omega, \mathbb{C})}^{\alpha} \langle M^*(z, w)e_i, e_k \rangle_{\ell^2} d\sigma(z) \\ &= \int_{\Omega} \sum_{i=1}^{\infty} \|K_z\|_{\mathcal{B}(\Omega, \mathbb{C})}^{\alpha} \left| \langle K_w e_k, T^*(K_z e_i) \rangle_{\mathcal{B}(\Omega)} \right| d\sigma(z) \\ &= \int_{\Omega} \sum_{i=1}^{\infty} \|K_z\|_{\mathcal{B}(\Omega, \mathbb{C})}^{\alpha} \left| \langle T(K_w e_k), K_z e_i \rangle_{\mathcal{B}(\Omega)} \right| d\sigma(z) \end{aligned}$$

$$\leq C \|K_w\|_{\mathcal{B}(\Omega, \mathbb{C})}^\alpha \sup_{e_k} \sup_{z \in \Omega} \left\{ \int_{\Omega} \left( \sum_{i=1}^{\infty} |\langle U_z T(k_z e_k)(u), e_i \rangle_{\ell^2}| \right)^p d\sigma(u) \right\}^{\frac{1}{p}}.$$

Thus, by the Matrix Schur's Test (Lemma 2.5) and our assumptions, the operator is bounded.  $\square$

**3.1. RKT for Toeplitz operators.** In the case when  $T = T_F$  is a Toeplitz operator, the conditions in Theorem 3.1 can be stated in terms of the symbol,  $F$ .

**Corollary 3.3.** *Let  $\mathcal{B}(\Omega)$  be a strong Bergman-type space. If  $\kappa > 0$  and  $T_F$  is a Toeplitz operator whose symbol  $F$  satisfies*

$$\sup_i \sup_{z \in \Omega} \sum_{k=1}^{\infty} \| \langle (F^* \circ \varphi_z) e_i, e_k \rangle_{\ell^2} \|_{L^p(\Omega, \mathbb{C}; d\sigma)} < \infty$$

and

$$\sup_k \sup_{z \in \Omega} \sum_{i=1}^{\infty} \| \langle (F \circ \varphi_z) e_k, e_i \rangle_{\ell^2} \|_{L^p(\Omega, \mathbb{C}; d\sigma)} < \infty,$$

for some  $p > \frac{4-\kappa}{2-\kappa}$  then  $T_F$  is bounded on  $\mathcal{B}(\Omega)$ .

*Proof.* We first show that for all finite  $e \in \ell^2$  there holds

$$| \langle (U_z T_F^* k_z e_i)(w), e_k \rangle_{\ell^2} | = | P( \langle (F^* \circ \varphi_z) e_i, e_k \rangle_{\ell^2} ) (w) |$$

and

$$| \langle (U_z T_F k_z e_k)(w), e_i \rangle_{\ell^2} | = | P( \langle (F \circ \varphi_z) e_k, e_i \rangle_{\ell^2} ) (w) |.$$

By A.5,  $|k_0| \equiv 1$  on  $\Omega$ . By the maximum and minimum modulus principles, this means that  $k_0$  is constant on  $\Omega$  and since  $k_0(0) = \|K_0\|_{\mathcal{B}(\Omega, \mathbb{C})} > 0$  there holds that  $k_0 \equiv 1$  on  $\Omega$ . Equation (2.4) will be used several times.

$$\begin{aligned} | \langle (U_z T_F k_z e_k)(w), e_i \rangle_{\ell^2} | &= \|K_w\|_{\mathcal{B}(\Omega, \mathbb{C})} \left| \langle U_z T_F k_z e_k, k_w e_i \rangle_{\mathcal{B}(\Omega)} \right| \\ &= \|K_w\|_{\mathcal{B}(\Omega, \mathbb{C})} \left| \int_{\Omega} \langle F(a) k_z(a) e_k, k_{\varphi_z(w)}(a) e_i \rangle_{\ell^2} d\sigma(a) \right| \\ &= \|K_w\|_{\mathcal{B}(\Omega, \mathbb{C})} \left| \int_{\Omega} \langle F(a) e_k, e_i \rangle_{\ell^2} \langle k_z, k_a \rangle_{\mathcal{B}(\Omega, \mathbb{C})} \overline{\langle k_{\varphi_z(w)}, k_a \rangle_{\mathcal{B}(\Omega, \mathbb{C})}} d\lambda(a) \right| \\ &= \|K_w\|_{\mathcal{B}(\Omega, \mathbb{C})} \left| \int_{\Omega} \langle F(\varphi_z(b)) e_k, e_i \rangle_{\ell^2} \langle k_z, k_{\varphi_z(b)} \rangle_{\mathcal{B}(\Omega, \mathbb{C})} \overline{\langle k_{\varphi_z(w)}, k_{\varphi_z(b)} \rangle_{\mathcal{B}(\Omega, \mathbb{C})}} d\lambda(b) \right| \\ &= \left| \int_{\Omega} \langle F(\varphi_z(b)) e_k, e_i \rangle_{\ell^2} k_0(b) \overline{\langle K_w, K_b \rangle_{\mathcal{B}(\Omega, \mathbb{C})}} d\lambda(b) \right| \\ &= | P( \langle (F \circ \varphi_z) e_k, e_i \rangle_{\ell^2} ) (w) |. \end{aligned}$$

And  $| \langle (U_z T_F^* k_z e_i)(w), e_k \rangle_{\ell^2} | = | P( \langle (F^* \circ \varphi_z) e_i, e_k \rangle_{\ell^2} ) (w) |$  is proven similarly.

Therefore, by the boundedness of the (scalar-valued) Bergman projection, Lemma 2.3, there holds:

$$\sup_i \sup_{z \in \Omega} \left\{ \int_{\Omega} \left( \sum_{k=1}^{\infty} | \langle U_z T_F^* (k_z e_i)(u), k \rangle_{\ell^2} | \right)^p d\sigma(u) \right\}^{\frac{1}{p}} \leq \sup_{e_i} \sup_{z \in \Omega} \sum_{k=1}^{\infty} \| \langle (F^* \circ \varphi_z) e_i, e_k \rangle_{\ell^2} \|_{L^p(\Omega, \mathbb{C}; d\sigma)}$$

and

$$\sup_{e_k} \sup_{z \in \Omega} \left\{ \int_{\Omega} \left( \sum_{i=1}^{\infty} |\langle U_z T_F(k_z e_k)(u), i \rangle_{\ell^2}| \right)^p d\sigma(u) \right\}^{\frac{1}{p}} \leq \sup_k \sup_{z \in \Omega} \sum_{i=1}^{\infty} \|\langle (F \circ \varphi_z) e_k, e_i \rangle_{\ell^2}\|_{L^p(\Omega, \mathbb{C}; d\sigma)}.$$

Therefore, the two conditions from Theorem 3.1 are satisfied and so  $T_F$  is bounded.  $\square$

**3.2. RKT for product of Toeplitz operators with analytic symbols.** In this section we derive a sufficient condition for boundedness of products Toeplitz operators,  $T_F T_{G^*}$ . For another result giving sufficient conditions for the boundedness of this product see [15].

**Corollary 3.4.** *Let  $\mathcal{B}(\Omega)$  be a strong  $\ell^2$ -valued Bergman-type space such that a product of any two reproducing kernels from  $\mathcal{B}(\Omega, \mathbb{C})$  is still in  $\mathcal{B}(\Omega, \mathbb{C})$ . Let  $F, G : \Omega \rightarrow \mathcal{L}(\ell^2)$  satisfy  $\langle F e_k, e_i \rangle_{\ell^2}, \langle G e_k, e_i \rangle_{\ell^2} \in \mathcal{B}(\Omega)$  for every  $i, k \in \mathbb{N}$ . If there exists  $p > \frac{4-\kappa}{2-\kappa}$  such that*

$$\sup_k \sup_{z \in \Omega} \sum_{i=1}^{\infty} \|\langle G^*(z) e_k, F^* \circ \varphi_z e_i \rangle_{\ell^2}\|_{L^p(\Omega, \mathbb{C}; d\sigma)} < \infty$$

and

$$\sup_i \sup_{z \in \Omega} \sum_{k=1}^{\infty} \|\langle F^*(z) e_i, G^* \circ \varphi_z e_k \rangle_{\ell^2}\|_{L^p(\Omega, \mathbb{C}; d\sigma)} < \infty$$

then the operator  $T_F T_{G^*}$  is bounded on  $\mathcal{B}(\Omega)$ .

*Proof.* We only need to check that  $T_F T_{G^*}$  satisfies the conditions of Theorem 3.1. We first show that  $\langle T_{G^*} k_z e_i, e_k \rangle_{\ell^2} = \langle G(z)^* k_z(w) e_i, e_k \rangle_{\ell^2}$ . First assume that  $\langle G^* e_i, e_k \rangle_{\ell^2}$  is a finite linear combination of reproducing kernels. Then  $K_w \langle G^* e_i, e_k \rangle_{\ell^2} = \langle K_w G^* e_i, e_k \rangle_{\ell^2} \in \mathcal{B}(\Omega, \mathbb{C})$  for any reproducing kernel  $K_w$ . Therefore,

$$\begin{aligned} \langle T_{G^*} k_z(w) e_i, e_k \rangle_{\ell^2} &= \int_{\Omega} \left\langle \langle K_u, K_w \rangle_{\mathcal{B}(\Omega, \mathbb{C})} G^*(u) k_z(u) e_i, e_k \right\rangle_{\ell^2} d\sigma(u) \\ &= \overline{\langle K_w G e_k, k_z e_i \rangle_{\mathcal{B}(\Omega)}} \\ &= \|K_z\|_{\mathcal{B}(\Omega, \mathbb{C})}^{-1} \overline{\langle K_w G e_k, K_z e_i \rangle_{\mathcal{B}(\Omega)}} \\ &= \langle G(z)^* k_z(w) e_i, e_k \rangle_{\ell^2}. \end{aligned}$$

Next, let  $G$  be arbitrary. Fix  $z, w \in \Omega$ . Let  $\epsilon > 0$ . There is a matrix-valued  $H : \Omega \rightarrow \mathcal{L}(\ell^2)$  such that  $\langle H e_i, e_k \rangle_{\ell^2}$  is a finite linear combination of reproducing kernels and  $\|\langle (G - H) e_i, e_k \rangle_{\ell^2}\|_{\mathcal{B}(\Omega, \mathbb{C})} < \epsilon$  and  $\|\langle (G - H) e_k, e_i \rangle_{\ell^2}\|_{\mathcal{B}(\Omega, \mathbb{C})} < \epsilon$ . That is,  $H$  is a matrix-valued function and the entries of  $H$  approximate the entries of  $G$ . Note that we are not claiming that  $H$  converges to  $G$  in any operator norm, this is only convergence in  $\mathcal{B}(\Omega, \mathbb{C})$  of the entries of  $H$  to the entries of  $G$ . Then there holds

$$\begin{aligned} |\langle T_{G^*} k_z(w) e_i, e_k \rangle_{\ell^2} - \langle H^*(z) k_z(w) e_i, e_k \rangle_{\ell^2}| &= |\langle T_{(G-H)^*} k_z(w) e_i, e_k \rangle_{\ell^2}| \\ &= \left| \int_{\Omega} \left\langle \langle K_u, K_w \rangle_{\mathcal{B}(\Omega, \mathbb{C})} ((G(u) - H(u))^*) k_z(u) e_i, e_k \right\rangle_{\ell^2} d\sigma(u) \right| \\ &= \left| \int_{\Omega} \overline{K_w(u) k_z(u)} \langle ((G(u) - H(u))^*) e_i, e_k \rangle_{\ell^2} d\sigma(u) \right| \\ &\leq \int_{\Omega} |K_w(u) k_z(u)|^2 d\sigma(u) \|\langle (G - H) e_k, e_i \rangle_{\ell^2}\|_{\mathcal{B}(\Omega, \mathbb{C})}^2 \end{aligned}$$

$$< C(z, w)\epsilon^2.$$

Moreover,

$$\begin{aligned} |\langle (G(z) - H(z))e_i, e_k \rangle_{\ell^2}| &= \left| \langle \langle (G(z) - H(z))e_i, e_k \rangle_{\ell^2}, K_z \rangle_{\mathcal{B}(\Omega, \mathbb{C})} \right| \\ &\leq \|K_z\|_{\mathcal{B}(\Omega, \mathbb{C})} \|\langle (G - H)e_i, e_k \rangle_{\ell^2}\|_{\mathcal{B}(\Omega, \mathbb{C})} \\ &< \|K_z\|_{\mathcal{B}(\Omega, \mathbb{C})} \epsilon. \end{aligned}$$

Since  $\epsilon > 0$  was arbitrary and  $z, w$  were fixed there holds  $\langle T_{G^*}k_z e_i, e_k \rangle_{\ell^2} = \langle G(z)^*k_z(w)e_i, e_k \rangle_{\ell^2}$  and  $\langle T_{F^*}k_z e_k, e_i \rangle_{\ell^2} = \langle F(z)^*k_z(w)e_k, e_i \rangle_{\ell^2}$ . It is also easy to see that this implies  $\langle T_{G^*}k_z e_i, f e_k \rangle_{\mathcal{B}(\Omega)} = \langle G(z)^*k_z e_i, f e_k \rangle_{\mathcal{B}(\Omega)}$  for  $f \in \mathcal{B}(\Omega, \mathbb{C})$ .

So, there holds:

$$\begin{aligned} |\langle U_z T_F T_{G^*} k_z e_k(w), e_i \rangle_{\ell^2}| &= \|K_w\|_{\mathcal{B}(\Omega, \mathbb{C})} \left| \langle U_z T_F T_{G^*} k_z e_k, k_w e_i \rangle_{\mathcal{B}(\Omega)} \right| \\ &= \|K_w\|_{\mathcal{B}(\Omega, \mathbb{C})} \left| \langle T_F T_G^*(z) k_z e_k, k_{\varphi_z(w)} e_i \rangle_{\mathcal{B}(\Omega)} \right| \\ &= \|K_w\|_{\mathcal{B}(\Omega, \mathbb{C})} \left| \langle G^*(z) k_z e_k, F^* \circ \varphi_z(w) k_{\varphi_z(w)} e_i \rangle_{\mathcal{B}(\Omega)} \right| \\ &= \|K_w\|_{\mathcal{B}(\Omega, \mathbb{C})} \left| \langle G^*(z) e_k, F^* \circ \varphi_z(w) e_i \rangle_{\ell^2} \langle k_z, k_{\varphi_z(w)} \rangle_{\mathcal{B}(\Omega, \mathbb{C})} \right| \\ &= |\langle G^*(z) e_k, F^* \circ \varphi_z(w) e_i \rangle_{\ell^2}|. \end{aligned}$$

Thus,

$$|\langle U_z T_F T_{G^*} k_z e_k(w), e_i \rangle_{\ell^2}| = |\langle G^*(z) e_k, F^* \circ \varphi_z(w) e_i \rangle_{\ell^2}|.$$

and

$$|\langle U_z T_G T_{F^*} k_z e_i(w), e_k \rangle_{\ell^2}| = |\langle F^*(z) e_i, G^* \circ \varphi_z(w) e_k \rangle_{\ell^2}|.$$

Using our hypotheses, we deduce that  $T_F T_{G^*}$  satisfies the conditions of Theorem 3.1.  $\square$

**3.3. RKT for Hankel operators.** Next we treat the case of Hankel operators. The Hankel operator  $H_F : \mathcal{B}(\Omega) \rightarrow \mathcal{B}(\Omega)^\perp$  with matrix-valued symbol  $F : \Omega \rightarrow \mathcal{L}(\ell^2)$  is defined by  $H_F g = (I - P)Fg$ , where  $P$  is the orthogonal projection of  $L^2(\Omega, \ell^2; d\sigma)$  onto  $\mathcal{B}(\Omega)$ . Since  $H_F$  is not a operator from  $\mathcal{B}(\Omega)$  to  $\mathcal{B}(\Omega)$ , we can't apply Theorem 3.1. However, we can reuse the proof to prove the following Corollary.

**Corollary 3.5.** *Let  $\mathcal{B}(\Omega)$  be a strong Bergman-type space. If  $H_F$  is a Hankel operator whose symbol  $F$  satisfies*

$$\sup_i \sup_{z \in \Omega} \left( \int_{\Omega} \left( \sum_{k=1}^{\infty} |\langle (F(z) - F(\varphi_z(u)))e_k, e_i \rangle_{\ell^2}| \right)^p d\sigma(u) \right)^{\frac{1}{p}} < \infty$$

for some  $p > \frac{4-\kappa}{2-\kappa}$  then  $H_F$  is bounded.

*Proof.* The proof is basically the same as for Theorem 3.1. As in the proof the Theorem 3.1, we show that there is a constant such that

$$\|H_F g\|_{\mathcal{B}(\Omega)} \lesssim \|g\|_{\mathcal{B}(\Omega)}$$

for any  $g \in \mathcal{B}(\Omega, \mathbb{C})$  that is a linear combination of normalized reproducing kernels. First, there holds:

$$\begin{aligned} (H_F g)(z) &= F(z)g(z) - P(Fg)(z) \\ &= \int_{\Omega} (F(z)g(w) - F(w)f(w)) \langle K_w, K_z \rangle_{\mathcal{B}(\Omega, \mathbb{C})} d\sigma(w) \\ &= \int_{\Omega} \sum_{i=1}^{\infty} \sum_{k=1}^{\infty} \langle (F(z) - F(w))e_k, e_i \rangle_{\ell^2} \langle K_w, K_z \rangle_{\mathcal{B}(\Omega, \mathbb{C})} g_k(w) e_i d\sigma(w) \end{aligned}$$

Thus, we want to show that the integral operator with matrix-valued kernel given by:

$$\langle M(z, w)e_k, e_i \rangle_{\ell^2} = |\langle (F(z) - F(w))e_k, e_i \rangle_{\ell^2}| \left| \langle K_z, K_w \rangle_{\mathcal{B}(\Omega, \mathbb{C})} \right|$$

is bounded. The Matrix Schur's Test, (Lemma 2.5), will be used to prove that the operator is bounded with

$$\langle M(z, w)e_k, e_i \rangle_{\ell^2} = |\langle (F(z) - F(w))e_k, e_i \rangle_{\ell^2}| \left| \langle K_z, K_w \rangle_{\mathcal{B}(\Omega, \mathbb{C})} \right|, \quad h(z) \equiv \|K_z\|_{\mathcal{B}(\Omega, \mathbb{C})}^{\alpha/2},$$

$$X = \Omega, \quad d\mu(z) = d\nu(z) = d\sigma(z).$$

If  $\kappa = 0$  set  $\alpha = \frac{4-2\kappa}{4-\kappa} = 1$ . If  $\kappa > 0$  choose  $\alpha \in (\frac{2}{p}, \frac{4-2\kappa}{4-\kappa})$  such that  $q(\alpha - \frac{2}{p}) < \kappa$ . The condition  $p > \frac{4-\kappa}{2-\kappa}$  ensures that such  $\alpha$  exists. Let  $z \in \Omega$  be arbitrary and fixed. There holds:

$$\begin{aligned} Q_1 &:= \int_{\Omega} \sum_{k=1}^{\infty} \|K_w\|_{\mathcal{B}(\Omega, \mathbb{C})}^{\alpha} \langle M(z, w)e_k, e_i \rangle_{\ell^2} d\sigma(w) \\ &= \int_{\Omega} \sum_{k=1}^{\infty} \|K_w\|_{\mathcal{B}(\Omega, \mathbb{C})}^{\alpha} |\langle (F(z) - F(w))e_k, e_i \rangle_{\ell^2}| \left| \langle K_z, K_w \rangle_{\mathcal{B}(\Omega, \mathbb{C})} \right| d\sigma(w) \\ &= \|K_z\|_{\mathcal{B}(\Omega, \mathbb{C})} \int_{\Omega} \sum_{k=1}^{\infty} |\langle (F(z) - F \circ \varphi_z(u))e_k, e_i \rangle_{\ell^2}| \left| \langle k_z, k_{\varphi_z(u)} \rangle_{\mathcal{B}(\Omega, \mathbb{C})} \right| \|K_{\varphi_z(u)}\|_{\mathcal{B}(\Omega, \mathbb{C})}^{\alpha-1} d\lambda(u) \\ &\simeq \|K_z\|_{\mathcal{B}(\Omega, \mathbb{C})} \int_{\Omega} \sum_{k=1}^{\infty} |\langle (F(z) - F \circ \varphi_z(u))e_k, e_i \rangle_{\ell^2}| \left| \langle k_z, U_z^* k_u \rangle_{\mathcal{B}(\Omega, \mathbb{C})} \right| \left| \langle k_z, k_u \rangle_{\mathcal{B}(\Omega, \mathbb{C})} \right|^{1-\alpha} d\lambda(u) \\ &= \|K_z\|_{\mathcal{B}(\Omega, \mathbb{C})} \int_{\Omega} \sum_{k=1}^{\infty} |\langle (F(z) - F \circ \varphi_z(u))e_k, e_i \rangle_{\ell^2}| \left| \langle U_z k_z, k_u \rangle_{\mathcal{B}(\Omega, \mathbb{C})} \right| \left| \langle k_z, k_u \rangle_{\mathcal{B}(\Omega, \mathbb{C})} \right|^{1-\alpha} d\lambda(u) \\ &= \|K_z\|_{\mathcal{B}(\Omega, \mathbb{C})}^{\alpha} \int_{\Omega} \sum_{k=1}^{\infty} |\langle (F(z) - F \circ \varphi_z(u))e_k, e_i \rangle_{\ell^2}| \frac{\left| \langle K_z, K_u \rangle_{\mathcal{B}(\Omega, \mathbb{C})} \right|^{1-\alpha}}{\|K_u\|_{\mathcal{B}(\Omega, \mathbb{C})}^{2-\alpha}} d\lambda(u). \end{aligned}$$

Using Holder's inequality we obtain that the last expression is no greater than

$$\|K_z\|_{\mathcal{B}(\Omega, \mathbb{C})}^{\alpha} \left( \int_{\Omega} \left( \sum_{k=1}^{\infty} |\langle (F(z) - F(\varphi_z(u)))e_k, e_i \rangle_{\ell^2}| \right)^p d\sigma(u) \right)^{\frac{1}{p}} \left( \int_{\Omega} \frac{|\langle K_z, K_u \rangle_{\mathcal{B}(\Omega, \mathbb{C})}|^{q(1-\alpha)}}{\|K_u\|_{\mathcal{B}(\Omega, \mathbb{C})}^{q(2-\alpha-\frac{2}{p})}} d\lambda(u) \right)^{\frac{1}{q}}.$$

Let  $r = q\left(2 - \alpha - \frac{2}{p}\right)$  and  $s = r - 2q(1 - \alpha)$ . Then  $r = \frac{p(2-\alpha-\frac{2}{p})}{p-1} = 2 - \frac{\alpha p}{p-1} > \kappa$  and  $s = q(\alpha - \frac{2}{p}) < \kappa$  when  $\kappa > 0$  and  $s = r > \kappa$  if  $\kappa = 0$ . This means that both  $r$  and  $s$  satisfy



all condition of A.6. Thus, by Lemma 2.6, the second integral is bounded independent of  $z$ . Call this constant  $C$ . This gives that:

$$Q_1 \leq \|K_z\|_{\mathcal{B}(\Omega, \mathbb{C})} C \sup_i \sup_{z \in \Omega} \left( \int_{\Omega} \left( \sum_{k=1}^{\infty} |\langle (F(z) - F(\varphi_z(u)))e_k, e_i \rangle_{\ell^2}| \right)^p d\sigma(u) \right)^{\frac{1}{p}}.$$

Now we check the second condition in Lemma 2.5.

$$\begin{aligned} Q_2 &:= \int_{\Omega} \sum_{i=1}^{\infty} \|K_z\|_{\mathcal{B}(\Omega, \mathbb{C})}^{\alpha} \langle M(z, w)^* e_i, e_k \rangle_{\ell^2} d\sigma(z) \\ &= \int_{\Omega} \sum_{i=1}^{\infty} \|K_z\|_{\mathcal{B}(\Omega, \mathbb{C})}^{\alpha} \langle e_i, M(z, w)e_k \rangle_{\ell^2} d\sigma(z) \\ &= \int_{\Omega} \sum_{i=1}^{\infty} \|K_z\|_{\mathcal{B}(\Omega, \mathbb{C})}^{\alpha} |\langle (F(z) - F(w))e_k, e_i \rangle_{\ell^2}| \left| \langle K_z, K_w \rangle_{\mathcal{B}(\Omega, \mathbb{C})} \right| d\sigma(w) \\ &= \int_{\Omega} \sum_{i=1}^{\infty} \|K_z\|_{\mathcal{B}(\Omega, \mathbb{C})}^{\alpha} |\langle e_k, (F(z) - F(w))^* e_i \rangle_{\ell^2}| \left| \langle K_z, K_w \rangle_{\mathcal{B}(\Omega, \mathbb{C})} \right| d\sigma(w). \end{aligned}$$

Using similar arguments as above, we conclude that:

$$\begin{aligned} Q_2 &\leq \|K_w\|_{\mathcal{B}(\Omega, \mathbb{C})} C \sup_k \sup_{z \in \Omega} \left( \int_{\Omega} \left( \sum_{i=1}^{\infty} |\langle (F(z) - F(\varphi_z(u)))^* e_i, e_k \rangle_{\ell^2}| \right)^p d\sigma(u) \right)^{\frac{1}{p}} \\ &= \|K_w\|_{\mathcal{B}(\Omega, \mathbb{C})} C \sup_k \sup_{z \in \Omega} \left( \int_{\Omega} \left( \sum_{i=1}^{\infty} |\langle (F(z) - F(\varphi_z(u)))e_k, e_i \rangle_{\ell^2}| \right)^p d\sigma(u) \right)^{\frac{1}{p}} \\ &= \|K_w\|_{\mathcal{B}(\Omega, \mathbb{C})} C \sup_i \sup_{z \in \Omega} \left( \int_{\Omega} \left( \sum_{k=1}^{\infty} |\langle (F(z) - F(\varphi_z(u)))e_k, e_i \rangle_{\ell^2}| \right)^p d\sigma(u) \right)^{\frac{1}{p}}. \end{aligned}$$

Thus, by our assumptions and the Matrix Schur's Test (Lemma 2.5), the operator is bounded.  $\square$

#### 4. REPRODUCING KERNEL THESES FOR COMPACTNESS

Compact operators on a Hilbert space are exactly the ones which send weakly convergent sequences into strongly convergent ones. In the current setting, there are, in essence, two “layers” of compactness that must be satisfied. For example, let  $\varphi$  be a scalar-valued function. Then the Toeplitz operator  $T_{\varphi I}$  is not compact on  $\mathcal{B}(\Omega)$  (unless  $\varphi \equiv 0$ ) since the sequence  $\{T_{\varphi I} e_k\}_{k=1}^{\infty}$  does not converge strongly to zero in  $\mathcal{B}(\Omega)$  but the sequence  $\{e_k\}_{k=1}^{\infty}$  converges weakly to 0 in  $\mathcal{B}(\Omega)$ . On the other hand, if  $T_{\varphi}$  is compact on  $\mathcal{B}(\Omega, \mathbb{C})$ , then  $T_{\varphi I(d)}$  is compact for any  $d \in \mathbb{N}$ .

The goal of this section is to prove that if  $T$  satisfies the conditions of Theorem 3.1 and another condition to be stated, and if  $T$  sends the weakly null sequence  $\{k_z e\}_{z \in \Omega}$  (see Lemma 4.1 below) into a strongly null sequence  $\{Tk_z e\}_{z \in \Omega}$ , then  $T$  must be compact.

Recall that the essential norm of a bounded linear operator  $S$  on  $\mathcal{B}(\Omega)$  is given by

$$\|S\|_e = \inf \left\{ \|S - A\|_{\mathcal{L}(\mathcal{B}(\Omega))} : A \text{ is compact on } \mathcal{B}(\Omega) \right\}.$$

We first show two simple results that will be used in the course of the proofs.

**Lemma 4.1.** *The weak limit of  $k_z e$  is zero as  $\mathfrak{d}(z, 0) \rightarrow \infty$ .*

*Proof.* Note first that property A.2 implies that if  $\mathfrak{d}(z, 0) \rightarrow \infty$  then  $\mathfrak{d}(\varphi_w(z), 0) \rightarrow \infty$ . Properties A.5 and A.7 now immediately imply that  $\langle k_w e, k_z h \rangle_{\mathcal{B}(\Omega)} \rightarrow 0$  as  $\mathfrak{d}(z, 0) \rightarrow \infty$ . The fact that the set  $\{k_z e_i : z \in \Omega, i \in \mathbb{N}\}$  is dense in  $\mathcal{B}(\Omega)$  then implies  $k_z e$  converges weakly to 0 as  $\mathfrak{d}(z, 0) \rightarrow \infty$ .  $\square$

**Lemma 4.2.** *For any compact operator  $A$  and any  $f \in \mathcal{B}(\Omega)$  we have that  $\|A^z f\|_{\mathcal{B}(\Omega)} \rightarrow 0$  as  $\mathfrak{d}(z, 0) \rightarrow \infty$ .*

*Proof.* If  $e \in \ell^2$  and  $f = k_w e$  then using the previous lemma we obtain that  $\|A^z k_w e\|_{\mathcal{B}(\Omega)} \simeq \|U_z A k_{\varphi_z(w)} e\|_{\mathcal{B}(\Omega)} \rightarrow 0$  as  $\mathfrak{d}(z, 0) \rightarrow \infty$ . For the general case, choose  $f \in \mathcal{B}(\Omega)$  arbitrary of norm 1. We can approximate  $f$  by linear combinations of normalized reproducing kernels and in a standard way we can deduce the same result.  $\square$

The following localization property will be a crucial step towards estimating the essential norm. A version of this result in the classical Bergman space setting was first proved by Suárez in [31]. Related results were later given in [4, 20, 22, 25].

**Proposition 4.3.** *Let  $T : \mathcal{B}(\Omega) \rightarrow \mathcal{B}(\Omega)$  be a linear operator and  $\kappa$  be the constant from A.6. If*

$$\sup_i \sup_{z \in \Omega} \left( \int_{\Omega} \left( \sum_{k=1}^{\infty} |\langle U_z T^* k_z e_i(u), e_k \rangle_{\ell^2}| \right)^p d\sigma(u) \right)^{\frac{1}{p}} < \infty$$

and

$$\sup_k \sup_{z \in \Omega} \left( \int_{\Omega} \left( \sum_{i=1}^{\infty} |\langle U_z T k_z e_k(u), e_i \rangle_{\ell^2}| \right)^p d\sigma(u) \right)^{\frac{1}{p}} < \infty$$

for some  $p > \frac{4-\kappa}{2-\kappa}$ , then for every  $\epsilon > 0$  there exists  $r > 0$  such that for the covering  $\mathcal{F}_r = \{F_j\}_{j=1}^{\infty}$  (associated to  $r$ ) from Proposition 2.9

$$\left\| T - \sum_j M_{1_{F_j}} T P M_{1_{G_j}} \right\|_{\mathcal{L}(\Omega, \ell^2; d\sigma)} < \epsilon.$$

*Proof.* Let  $r > 0$  and let  $\{F_j\}_{j=1}^{\infty}$  and  $\{G_j\}_{j=1}^{\infty}$  be the sets from Proposition 2.9 for this value of  $r$ . Let  $f \in \mathcal{B}(\Omega)$  have norm at most 1 there holds:

$$\begin{aligned} (Tf)(z) - \sum_{j=1}^{\infty} (M_{1_{F_j}} T P M_{1_{G_j}} f)(z) &= \sum_{j=1}^{\infty} M_{1_{F_j}} (Tf - T P M_{1_{G_j}} f)(z) \\ &= \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} 1_{F_j}(z) \langle Tf - T 1_{G_j} f, K_z e_i \rangle_{\mathcal{B}(\Omega)} e_i \\ &= \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} 1_{F_j}(z) \langle f - 1_{G_j} f, T^* K_z e_i \rangle_{\mathcal{B}(\Omega)} e_i \end{aligned}$$

$$\begin{aligned}
&= \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} 1_{F_j}(z) \left\langle 1_{G_j^c} f, T^* K_z e_i \right\rangle_{\mathcal{B}(\Omega)} e_i \\
&= \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} \int_{\Omega} 1_{F_j}(z) 1_{G_j^c}(w) \langle f(w), T^*(K_z e_i)(w) \rangle_{\ell^2} d\sigma(w) e_i \\
&= \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} \int_{\Omega} 1_{F_j}(z) 1_{G_j^c}(w) \sum_{k=1}^{\infty} f_k(w) \langle e_k, T^*(K_z e_i)(w) \rangle_{\ell^2} d\sigma(w) e_i \\
&= \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} \int_{\Omega} 1_{F_j}(z) 1_{G_j^c}(w) \sum_{k=1}^{\infty} f_k(w) \langle K_w e_k, T^*(K_z e_i) \rangle_{\mathcal{B}(\Omega)} d\sigma(w) e_i.
\end{aligned}$$

Thus, we want to show that the integral operator with kernel given by:

$$\langle M_r(z, w) e_k, e_i \rangle_{\ell^2} = \sum_{j=1}^{\infty} 1_{F_j}(z) 1_{G_j^c}(w) \left| \langle T^* K_z e_i, K_w e_k \rangle_{\mathcal{B}(\Omega)} \right|$$

is bounded and that the operator norm goes to zero as  $r \rightarrow \infty$ .

Again we will use the Matrix Schur's Test (Lemma 2.5) with

$$\langle M_r(z, w) e_k, e_i \rangle_{\ell^2} = \sum_{j=1}^{\infty} 1_{F_j}(z) 1_{G_j^c}(w) \left| \langle T^* K_z e_i, K_w e_k \rangle_{\mathcal{B}(\Omega)} \right|, \quad h(z) \equiv \|K_z\|_{\mathcal{B}(\Omega, \mathbb{C})}^{\alpha/2},$$

$$X = \Omega, \quad d\mu(z) = d\nu(z) = d\sigma(z).$$

If  $\kappa = 0$  then set  $\alpha = \frac{4-2\kappa}{4-\kappa} = 1$ . If  $\kappa > 0$  first choose  $p_0$  such that  $\frac{2}{p} < \frac{2}{p_0} < \frac{4-2\kappa}{4-\kappa}$  and denote by  $q_0$  the conjugate of  $p_0$ ,  $q_0 = \frac{p_0}{p_0-1}$ . Then choose  $\alpha \in (\frac{2}{p_0}, \frac{4-2\kappa}{4-\kappa})$  such that  $q_0(\alpha - \frac{2}{p_0}) < \kappa$ . The condition  $p > \frac{4-\kappa}{2-\kappa}$  ensures that such  $p_0$  and  $\alpha$  exist. Let  $z \in \Omega$  be arbitrary and fixed. Since  $\{F_j\}_{j=1}^{\infty}$  forms a covering for  $\Omega$  there exists a unique  $j$  such that  $z \in F_j$ . Note also that  $D(z, r) \subset G_j$  so  $G_j^c \subset D(z, r)^c$ . There holds:

$$\begin{aligned}
Q_1(r) &:= \int_{\Omega} \|K_w\|_{\mathcal{B}(\Omega, \mathbb{C})}^{\alpha} \sum_{k=1}^{\infty} \langle M_r(x, y) e_k, e_i \rangle_{\ell^2} d\sigma(w) \\
&= \int_{\Omega} \|K_w\|_{\mathcal{B}(\Omega, \mathbb{C})}^{\alpha} \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} 1_{F_j}(z) 1_{G_j^c}(w) \left| \langle T^* K_z e_i, K_w e_k \rangle_{\mathcal{B}(\Omega)} \right| d\sigma(w) \\
&= \int_{G_j^c} \|K_w\|_{\mathcal{B}(\Omega, \mathbb{C})}^{\alpha} 1_{F_j}(z) \sum_{k=1}^{\infty} \left| \langle T^* K_z e_i, K_w e_k \rangle_{\mathcal{B}(\Omega)} \right| d\sigma(w) \\
&\leq \int_{D(z, r)^c} \|K_w\|_{\mathcal{B}(\Omega, \mathbb{C})}^{\alpha} 1_{F_j}(z) \sum_{k=1}^{\infty} \left| \langle T^* K_z e_i, K_w e_k \rangle_{\mathcal{B}(\Omega)} \right| d\sigma(w) \\
&= \|K_z\|_{\mathcal{B}(\Omega, \mathbb{C})} \int_{D(0, r)^c} \sum_{k=1}^{\infty} \left| \langle T^* k_z e_i, k_{\varphi_z(u)} e_k \rangle_{\mathcal{B}(\Omega)} \right| \|K_{\varphi_z(u)}\|_{\mathcal{B}(\Omega, \mathbb{C})-1}^{\alpha-1} d\lambda(u) \\
&\simeq \|K_z\|_{\mathcal{B}(\Omega, \mathbb{C})} \int_{D(0, r)^c} \sum_{k=1}^{\infty} \left| \langle T^* U_z^* k_0 e_i, U_z^* k_u e_k \rangle_{\mathcal{B}(\Omega)} \right| \left| \langle k_z, k_u \rangle_{\mathcal{B}(\Omega, \mathbb{C})} \right|^{1-\alpha} d\lambda(u)
\end{aligned}$$

$$\begin{aligned}
&= \|K_z\|_{\mathcal{B}(\Omega, \mathbb{C})} \int_{D(0, r)^c} \sum_{k=1}^{\infty} \left| \langle T^{*z} k_0 e_i, k_u e_k \rangle_{\mathcal{B}(\Omega)} \right| \left| \langle k_z, k_u \rangle_{\mathcal{B}(\Omega, \mathbb{C})} \right|^{1-\alpha} d\lambda(u) \\
&= \|K_z\|_{\mathcal{B}(\Omega, \mathbb{C})}^{\alpha} \int_{D(0, r)^c} \sum_{k=1}^{\infty} \left| \langle T^{*z} (k_0 e_i)(u), e_k \rangle_{\ell^2} \right| \frac{\left| \langle K_z, K_u \rangle_{\mathcal{B}(\Omega, \mathbb{C})} \right|^{1-\alpha}}{\|K_u\|_{\mathcal{B}(\Omega, \mathbb{C})}^{2-\alpha}} d\lambda(u).
\end{aligned}$$

Using Hölder's inequality we obtain that the last expression is no greater than

$$\|K_z\|_{\mathcal{B}(\Omega, \mathbb{C})}^{\alpha} \left( \int_{D(0, r)^c} \left( \sum_{k=1}^{\infty} \left| \langle T^{*z} k_0 e_i(u), e_k \rangle_{\ell^2} \right| \right)^{p_0} d\sigma(u) \right)^{\frac{1}{p_0}} \left( \int_{\Omega} \frac{\left| \langle K_z, K_u \rangle_{\mathcal{B}(\Omega, \mathbb{C})} \right|^{q_0(1-\alpha)}}{\|K_u\|_{\mathcal{B}(\Omega, \mathbb{C})}^{q_0(2-\alpha-\frac{2}{p_0})}} d\lambda(u) \right)^{\frac{1}{q_0}}.$$

Let  $r = q_0 \left( 2 - \alpha - \frac{2}{p_0} \right)$  and  $s = r - 2q_0(1 - \alpha)$ . Then  $r = \frac{p_0(2-\alpha-\frac{2}{p_0})}{p_0-1} = 2 - \frac{\alpha p_0}{p_0-1} > \kappa$  and  $s = q_0(\alpha - \frac{2}{p_0}) < \kappa$  when  $\kappa > 0$  and  $s = r > \kappa$  if  $\kappa = 0$ . This means that both  $r$  and  $s$  satisfy all conditions of [A.6](#). Thus, by Lemma [2.6](#), the second integral is bounded independent of  $z$ . Call this constant  $C$ .

For the first integral, note that  $|\langle T^{*z} k_0 e_i(u), e_k \rangle_{\ell^2}| \simeq |\langle U_z T^* k_z e_i(u), e_k \rangle_{\ell^2}|$ . Then using Hölder's Inequality there holds:

$$Q_1(r) \lesssim \|K_z\|_{\mathcal{B}(\Omega, \mathbb{C})}^{\alpha} C \sigma(D(0, r)^c)^{\gamma} \left( \int_{\Omega} \left( \sum_{k=1}^{\infty} |\langle U_z T^* k_z e_i(u), e_k \rangle_{\ell^2}| \right)^p d\sigma(u) \right)^{\frac{1}{p}}.$$

Where  $\gamma = 1/p_0 p'$  and  $p'$  is conjugate exponent to  $p$ . Since  $\sigma$  is a finite measure and since

$$\sup_i \sup_{z \in \Omega} \left( \int_{\Omega} \left( \sum_{k=1}^{\infty} |\langle U_z T^* k_z e_i(u), e_k \rangle_{\ell^2}| \right)^p d\sigma(u) \right)^{\frac{1}{p}} < \infty,$$

$Q_1(r)$  goes to 0 as  $r \rightarrow \infty$ . Thus, the first condition of Lemma [2.5](#) is satisfied with a constant  $o(1)$  as  $r \rightarrow \infty$ .

Next, we check the second condition. Fix  $w \in \Omega$ . Let  $J$  be a subset of all indices  $j$  such that  $w \notin G_j$ . If  $z \in F_j$  for some  $j \in J$ , then since  $w \notin G_j$ , there holds that  $\mathfrak{d}(w, F_j) > r$  and therefore  $z$  is not in  $D(w, r)$ . Thus,  $\cup_{j \in J} F_j \subset D(w, r)^c$  and consequently

$$\begin{aligned}
Q_2(r) &:= \int_{\Omega} \sum_{i=1}^{\infty} \|K_z\|_{\mathcal{B}(\Omega, \mathbb{C})} \langle M_r^*(z, w) e_i, e_k \rangle_{\ell^2} d\sigma(z) \\
&= \int_{\Omega} \sum_{i=1}^{\infty} \|K_z\|_{\mathcal{B}(\Omega, \mathbb{C})} \langle e_i, M_r(z, w) e_k \rangle_{\ell^2} d\sigma(z) \\
&= \int_{\Omega} \sum_{i=1}^{\infty} \|K_z\|_{\mathcal{B}(\Omega, \mathbb{C})} \sum_{j=1}^{\infty} 1_{F_j}(z) 1_{G_j^c}(w) \left| \langle T^* K_z e_i, K_w e_k \rangle_{\mathcal{B}(\Omega)} \right| d\sigma(z) \\
&= \int_{\Omega} \|K_z\|_{\mathcal{B}(\Omega, \mathbb{C})} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} 1_{F_j}(z) 1_{G_j^c}(w) \left| \langle K_z e_i, T K_w e_k \rangle_{\mathcal{B}(\Omega)} \right| d\sigma(z) \\
&= \int_{\cup_{j \in J} F_j} \|K_z\|_{\mathcal{B}(\Omega, \mathbb{C})} \sum_{i=1}^{\infty} \left| \langle K_z e_i, T K_w e_k \rangle_{\mathcal{B}(\Omega)} \right| d\sigma(z)
\end{aligned}$$

$$\leq \int_{D(w,r)^c} \|K_z\|_{\mathcal{B}(\Omega, \mathbb{C})} \sum_{i=1}^{\infty} \left| \langle K_z e_i, T K_w e_k \rangle_{\mathcal{B}(\Omega)} \right| d\sigma(z).$$

Now, using the same estimates as above, but interchanging roles of  $T$  and  $T^*$  and  $i$  and  $k$  there holds:

$$Q_2(r) \lesssim \|K_w\|_{\mathcal{B}(\Omega, \mathbb{C})}^\alpha C\sigma(D(0,r)^c)^\gamma \left( \int_{\Omega} \left( \sum_{i=1}^{\infty} |\langle U_z T k_z e_k(u), e_i \rangle_{\ell^2}| \right)^p d\sigma(u) \right)^{\frac{1}{p}}.$$

Thus, as before,  $Q_2(r)$  goes to zero as  $r \rightarrow \infty$ .

So both conditions of Matrix Schur's Test (Lemma 2.5) are satisfied with constants that go to zero as  $r \rightarrow \infty$ . Thus, by choosing  $r$  large enough, the integral operator with kernel given by  $M_r(z, w)$  has operator norm less than  $\epsilon$ . If  $\{F_j\}_{j=1}^{\infty}$  and  $\{G_j\}_{j=1}^{\infty}$  are the sets from Proposition 2.9 associated to this valued of  $r$ , this also implies that:

$$\left\| T - \sum_{j=1}^{\infty} M_{1_{F_j}} T P M_{1_{G_j}} \right\|_{\mathcal{L}(\Omega, \ell^2; d\sigma)} < \epsilon.$$

This proves the proposition for  $\kappa > 0$ . When  $\kappa = 0$ , the Proposition can be proven by making adaptations as in the proof of Theorem 3.1.  $\square$

We now come to the main results of the section.

**Theorem 4.4.** *Let  $T : \mathcal{B}(\Omega) \rightarrow \mathcal{B}(\Omega)$  be a linear operator and  $\kappa$  be the constant from A.6. If*

$$\sup_i \sup_{z \in \Omega} \left( \int_{\Omega} \left( \sum_{k=1}^{\infty} |\langle U_z T^* k_z e_i(u), e_k \rangle_{\ell^2}| \right)^p d\sigma(u) \right)^{\frac{1}{p}} < \infty \quad (4.1)$$

and

$$\sup_k \sup_{z \in \Omega} \left( \int_{\Omega} \left( \sum_{i=1}^{\infty} |\langle U_z T k_z e_k(u), e_i \rangle_{\ell^2}| \right)^p d\sigma(u) \right)^{\frac{1}{p}} < \infty \quad (4.2)$$

for some  $p > \frac{4-\kappa}{2-\kappa}$ , and

$$\limsup_{d \rightarrow \infty} \|T M_{I(d)}\|_{\mathcal{L}(\mathcal{B}(\Omega), L^2(\Omega, \ell^2; d\sigma))} = 0 \quad (4.3)$$

then

- (a)  $\|T\|_e \simeq \sup_{\|f\| \leq 1} \limsup_{\mathfrak{d}(z,0) \rightarrow \infty} \|T^z f\|_{\mathcal{B}(\Omega)}.$
- (b) *If  $\sup_{e \in \mathbb{C}^d, \|e\|_{\mathbb{C}^d}=1} \lim_{\mathfrak{d}(z,0) \rightarrow \infty} \|T k_z e\|_{\mathcal{B}(\Omega, \mathbb{C})} = 0$  then  $T$  must be compact.*

*Proof.* We first prove (a). It is easy to deduce that

$$\sup_{\|f\|_{\mathcal{B}(\Omega)} \leq 1} \limsup_{\mathfrak{d}(z,0) \rightarrow \infty} \|T^z f\|_{\mathcal{B}(\Omega)} \lesssim \|T\|_e. \quad (4.4)$$

Indeed, using the triangle inequality and the fact that  $\lim_{\mathfrak{d}(z,0) \rightarrow \infty} \|A^z f\|_{\mathcal{B}(\Omega, \mathbb{C})} = 0$  for every compact operator  $A$  (Lemma 4.2) we obtain that

$$\sup_{\|f\|_{\mathcal{B}(\Omega)} \leq 1} \limsup_{\mathfrak{d}(z,0) \rightarrow \infty} \|T^z f\|_{\mathcal{B}(\Omega)} \leq \sup_{\|f\|_{\mathcal{B}(\Omega)} \leq 1} \limsup_{\mathfrak{d}(z,0) \rightarrow \infty} \|(T - A)^z f\|_{\mathcal{B}(\Omega)} \lesssim \|T - A\|_{\mathcal{L}(\mathcal{B}(\Omega), L^2(\Omega, \ell^2; d\sigma))}$$

for any compact operator  $A$ . Now, since  $A$  is arbitrary this immediately implies (4.4).

The other inequality requires more work. Proposition 4.3 and assumption (4.3) will play prominent roles. Observe that the essential norm of  $T$  as an operator in  $\mathcal{L}(\mathcal{B}(\Omega))$  is quasi-equal to the essential norm of  $T$  as an operator in  $\mathcal{L}(\mathcal{B}(\Omega), L^2(\Omega, \ell^2; d\sigma))$ . Therefore, it is enough to estimate the essential norm of  $T$  as an operator on  $\mathcal{L}(\mathcal{B}(\Omega), L^2(\Omega, \ell^2; d\sigma))$ .

Let  $\epsilon > 0$  and fix a  $d$  so large that

$$\left\| TM_{I^{(d)}} \right\|_{\mathcal{L}(\mathcal{B}(\Omega), L^2(\Omega, \ell^2; d\sigma))} < \epsilon.$$

Then

$$\|T\|_e = \left\| TM_{I^{(d)}} + TM_{I^{(d)}} \right\|_e \leq \|TM_{I^{(d)}}\|_e + \epsilon.$$

By Proposition 4.3 there exists  $r > 0$  such that for the covering  $\mathcal{F}_r = \{F_j\}_{j=1}^\infty$  associated to  $r$

$$\left\| TM_{I^{(d)}} - \sum_{j=1}^\infty M_{1_{F_j}} T P M_{1_{G_j}} M_{I^{(d)}} \right\|_{\mathcal{L}(\mathcal{B}(\Omega), L^2(\Omega, \ell^2; d\sigma))} < \epsilon.$$

Note that by Lemma 2.8 the Toeplitz operators  $P M_{1_{G_j}} M_{I^{(d)}}$  are compact. Therefore the finite sum  $\sum_{j \leq m} M_{1_{F_j}} T P M_{1_{G_j}} M_{I^{(d)}}$  is compact for every  $m, d \in \mathbb{N}$ . So, it is enough to show that

$$\limsup_{m \rightarrow \infty} \|T_{(m,d)}\|_{\mathcal{L}(\mathcal{B}(\Omega), L^2(\Omega, \ell^2; d\sigma))} \lesssim \sup_{\|f\| \leq 1} \limsup_{\mathfrak{d}(z,0) \rightarrow \infty} \|T^z f\|_{\mathcal{B}(\Omega)},$$

where

$$T_{(m,d)} = \sum_{j \geq m} M_{1_{F_j}} T P M_{1_{G_j}} M_{I^{(d)}}.$$

Indeed,

$$\begin{aligned} \|TM_{I^{(d)}}\|_e &= \|TM_{I^{(d)}}P\|_e \\ &\leq \left\| T P M_{I^{(d)}} - \sum_{j \leq m} M_{1_{F_j}} T P M_{1_{G_j}} M_{I^{(d)}} \right\|_{\mathcal{L}(\mathcal{B}(\Omega), L^2(\Omega, \ell^2; d\sigma))} \\ &\leq \epsilon + \|T_{(m,d)}\|_{\mathcal{L}(\mathcal{B}(\Omega), L^2(\Omega, \ell^2; d\sigma))}. \end{aligned}$$

Of course, the implied constants should be independent of the truncation parameter,  $d$ . Let  $f \in \mathcal{B}(\Omega)$  be arbitrary of norm no greater than 1. There holds:

$$\begin{aligned} \|T_{(m,d)}f\|_{\mathcal{L}(\mathcal{B}(\Omega), L^2(\Omega, \ell^2; d\sigma))}^2 &= \sum_{j \geq m} \left\| M_{1_{F_j}} T P M_{1_{G_j}} M_{I^{(d)}} f \right\|_{\mathcal{B}(\Omega)}^2 \\ &= \sum_{j \geq m} \frac{\left\| M_{1_{F_j}} T P M_{1_{G_j}} M_{I^{(d)}} f \right\|_{\mathcal{B}(\Omega)}^2}{\left\| M_{1_{G_j}} M_{I^{(d)}} f \right\|_{\mathcal{B}(\Omega)}^2} \left\| M_{1_{G_j}} M_{I^{(d)}} f \right\|_{\mathcal{B}(\Omega)}^2 \\ &\leq N \sup_{j \geq m} \left\| M_{1_{F_j}} T l_j \right\|_{\mathcal{B}(\Omega)}^2 \\ &\leq N \sup_{j \geq m} \|T l_j\|_{\mathcal{B}(\Omega)}^2, \end{aligned}$$

where

$$l_j := \frac{PM_{1_{G_j}}M_{I^{(d)}}f}{\|M_{1_{G_j}}M_{I^{(d)}}f\|_{\mathcal{B}(\Omega)}}.$$

Therefore,

$$\|T_{(m,d)}\|_{\mathcal{L}(\mathcal{B}(\Omega), L^2(\Omega, \ell^2; d\sigma))} \leq N \sup_{j \geq m} \sup_{\|f\|_{\mathcal{B}(\Omega)}=1} \left\{ \|Tl_j\|_{\mathcal{B}(\Omega)} : l_j = \frac{PM_{1_{G_j}}M_{I^{(d)}}f}{\|M_{1_{G_j}}M_{I^{(d)}}f\|_{\mathcal{B}(\Omega)}} \right\},$$

and hence

$$\limsup_{m \rightarrow \infty} \|T_{(m,d)}\|_{\mathcal{L}(\mathcal{B}(\Omega), L^2(\Omega, \ell^2; d\sigma))} \leq N \limsup_{j \rightarrow \infty} \sup_{\|f\|_{\mathcal{B}(\Omega)}=1} \left\{ \|Tg\| : g = \frac{PM_{1_{G_j}}M_{I^{(d)}}f}{\|M_{1_{G_j}}M_{I^{(d)}}f\|_{\mathcal{B}(\Omega)}} \right\}.$$

Let  $\epsilon > 0$ . There exists a normalized sequence  $\{f_j\}_{j=1}^\infty$  in  $\mathcal{B}(\Omega)$  such that

$$\limsup_{j \rightarrow \infty} \sup_{\|f\|_{\mathcal{B}(\Omega)}=1} \left\{ \|Tg\| : g = \frac{PM_{1_{G_j}}M_{I^{(d)}}f}{\|M_{1_{G_j}}M_{I^{(d)}}f\|_{\mathcal{B}(\Omega)}} \right\} - \epsilon \leq \limsup_{j \rightarrow \infty} \|Tg_j\|_{\mathcal{B}(\Omega)},$$

where

$$g_j := \frac{PM_{1_{G_j}}M_{I^{(d)}}f_j}{\|M_{1_{G_j}}M_{I^{(d)}}f_j\|_{\mathcal{B}(\Omega)}} = \frac{\int_{G_j} \sum_{k=1}^d \langle f_j, k_w e_k \rangle_{\mathcal{B}(\Omega)} k_w e_k d\lambda(w)}{\left( \int_{G_j} \sum_{k=1}^d \left| \langle f_j, k_w e_k \rangle_{\mathcal{B}(\Omega)} \right|^2 d\lambda(w) \right)^{\frac{1}{2}}}.$$

It is clear that the functions  $g_j$  are  $d$ -finite. Recall that  $|U_z^* k_w| \simeq |k_{\varphi_z(w)}|$ , and therefore,  $U_z^* k_w = c(w, z) k_{\varphi_z(w)}$ , where  $c(w, z)$  is some function so that  $|c(w, z)| \simeq 1$ .

There exists a  $\rho > 0$  such that if  $z_j \in G_j$  then  $G_j \subset D(z_j, \rho)$ . Thus, for each  $j$ , choose a  $z_j$  in  $G_j$ . By a change of variables, there holds

$$g_j = \int_{\varphi_{z_j}(G_j)} a_j(\varphi_{z_j}(w)) U_{z_j}^* k_w d\lambda(\varphi_{z_j}(w)),$$

where

$$a_j(w) := \frac{\sum_{k=1}^d \langle f_j, k_w e_k \rangle_{\mathcal{B}(\Omega)} e_k}{c(\varphi_{z_j}(w), z_j) \left( \int_{G_j} \sum_{k=1}^d \left| \langle f_j, k_w e_k \rangle_{\mathcal{B}(\Omega)} \right|^2 d\lambda(w) \right)^{\frac{1}{2}}}$$

on  $G_j$ , and zero otherwise.

We claim that  $g_j = U_{z_j}^* h_j$ , where

$$h_j(z) := \int_{\varphi_{z_j}(G_j)} a_j(\varphi_{z_j}(w)) k_w(z) d\lambda(\varphi_{z_j}(w)).$$

First, by applying the integral form of Minkowski's inequality to the components of  $h_j$ , we conclude that each component is in  $L^2(\Omega, \ell^2; \sigma)$  and therefore  $h_j$  is also in  $L^2(\Omega, \ell^2; \sigma)$ , and consequently in  $\mathcal{B}(\Omega)$ . Now we need to show that for every  $g \in L^2(\Omega, \ell^2; \sigma)$  there holds



$\langle g_j, g \rangle_{L^2(\Omega, \ell^2; \sigma)} = \langle U_{z_j}^* h_j, g \rangle_{L^2(\Omega, \ell^2; \sigma)} = \langle h_j, U_{z_j} g \rangle_{L^2(\Omega, \ell^2; \sigma)}$ . This is done by applying Fubini's Theorem component-wise.

For each  $k = 1, \dots, d$ , the total variation of each member of the sequence of measures  $\{\langle a_j \circ \varphi_{z_j}, e_k \rangle_{\ell^2} d\lambda \circ \varphi_{z_j}\}_{j=1}^\infty$ , as elements in the dual space of continuous functions on  $(\overline{D(0, \rho)})$  satisfies

$$\|\langle a_j \circ \varphi_{z_j}, e_k \rangle_{\ell^2} d\lambda \circ \varphi_{z_j}\|_{C(\overline{D(0, \rho)})^*} \lesssim \lambda(D(0, \rho)).$$

Therefore, for each  $k$ , there exists a weak-\* convergent subsequence which approaches some measure  $\nu_k$ . Let

$$h(z) = \sum_{k=1}^d \int_{D(0, \rho)} k_w(z) d\nu_k(w) e_k.$$

Abusing notation, we continue to index the subsequence by  $j$ . The weak-\* convergence implies that  $\langle h_j, e_k \rangle_{\ell^2}$  converges to  $\langle h, e_k \rangle_{\ell^2}$  pointwise. By the Lebesgue Dominated Convergence Theorem, this implies that  $\langle h_j, e_k \rangle_{\ell^2}$  converges to  $\langle h, e_k \rangle_{\ell^2}$  in  $L^2(\Omega, \mathbb{C}; \sigma)$  and thus  $\langle h, e_k \rangle_{\ell^2}$  is in  $L^2(\Omega, \mathbb{C}; d\sigma)$ . Since the  $h_j$  and  $h$  are  $d$ -finite, this implies that  $h_j$  converges to  $h$  in  $L^2(\Omega, \ell^2; d\sigma)$  and that  $h \in L^2(\Omega, \ell^2; d\sigma)$ . Additionally,  $1 \geq \|g_j\|_{\mathcal{B}(\Omega)} = \|U_{z_j}^* h_j\|_{\mathcal{B}(\Omega)} \simeq \|h_j\|_{\mathcal{B}(\Omega)}$ . So,  $\|h\|_{\mathcal{B}(\Omega)} \lesssim 1$ . So, finally, there holds:

$$\begin{aligned} \limsup_{m \rightarrow \infty} \|T_m\|_{\mathcal{L}(\mathcal{B}(\Omega), L^2(\Omega, \ell^2; d\sigma))} &\leq N \lim_{j \rightarrow \infty} \|Tg_j\|_{L^2(\Omega, \ell^2; d\sigma)} + \epsilon \\ &= N \lim_{j \rightarrow \infty} \|TU_{z_j}^* h_j\|_{L^2(\Omega, \ell^2; \sigma)} + \epsilon \\ &\leq N \limsup_{j \rightarrow \infty} \|TU_{z_j}^* h\|_{L^2(\Omega, \ell^2; \sigma)} + \epsilon \\ &\lesssim N \limsup_{j \rightarrow \infty} \|T^{z_j} h\|_{L^2(\Omega, \ell^2; \sigma)} + \epsilon. \end{aligned}$$

Again, the constants of equivalency do not depend on  $d$ . Therefore,

$$\limsup_{m \rightarrow \infty} \|T_m\|_{\mathcal{L}(\mathcal{B}(\Omega), L^2(\Omega, \ell^2; d\sigma))} \lesssim \sup_{\|f\|_{L^2(\Omega, \ell^2; \sigma)} \leq 1} \limsup_{d(z, 0) \rightarrow \infty} \|T^z f\|_{L^2(\Omega, \ell^2; \sigma)}.$$

(b) Note that  $\|T^z k_w e\|_{\mathcal{B}(\Omega)} \simeq \|Tk_{\varphi_z(w)} e\|_{\mathcal{B}(\Omega)}$  and  $d(\varphi_z(w), 0) \simeq d(w, z) \rightarrow \infty$  as  $d(z, 0) \rightarrow \infty$ . Therefore, for all  $w \in \Omega$  and finite  $e \in \ell^2$   $\|T^z k_w e\|_{\mathcal{B}(\Omega)} \rightarrow 0$  as  $d(z, 0) \rightarrow \infty$ . By density, this implies that  $\|T\|_e \simeq \sup_{\|f\|_{L^2(\Omega, \ell^2; \sigma)} \leq 1} \limsup_{d(z, 0) \rightarrow \infty} \|T^z f\|_{L^2(\Omega, \ell^2; \sigma)} = 0$ . We are done.  $\square$

**Corollary 4.5.** *Let  $\mathcal{B}(\Omega)$  be a strong Bergman-type space for which  $\kappa > 0$ . If  $T$  is in the Toeplitz algebra  $\mathcal{T}_{\text{fin}}^\infty$  then*

- (a)  $\|T\|_e \simeq \sup_{\|f\|_{\mathcal{B}(\Omega)} \leq 1} \limsup_{\mathfrak{d}(z, 0) \rightarrow \infty} \|T^z f\|_{\mathcal{B}(\Omega)}$ .
- (b) *If  $\sup_{e \in \mathbb{C}^d, \|e\|_{\mathbb{C}^d} = 1} \lim_{\mathfrak{d}(z, 0) \rightarrow \infty} \|Tk_z e\|_{\mathcal{B}(\Omega)} = 0$  then  $T$  must be compact.*

*Proof.* We will show that  $T$  satisfies the hypotheses of Theorem 4.4. First, let

$$T = \sum_{k=1}^M \prod_{j=1}^N T_{u_{j,k}}$$

where each  $u_{j,k} \in L_{\text{fin}}^\infty$  and is  $d_{j,k}$ -finite. By the triangle inequality, it suffices to show that  $T$  satisfies the hypotheses of Theorem 4.4 when  $T = \prod_{j=1}^N T_{u_j}$  and  $u_j \in L_{\text{fin}}^\infty$  and  $u_j$  is  $d_j$ -finite. Clearly,  $T$  satisfies (4.3). Now we will show that it also satisfies (4.1) and (4.2). For any  $z \in \Omega$  and  $i, k \in \mathbb{N}$  there holds

$$\begin{aligned} \langle U_z T k_z e_k, e_i \rangle_{\ell^2} &= \left\langle \left( \prod_{j=1}^N T_{u_j \circ \varphi_z} \right) (k_0 e_k), e_i \right\rangle_{\ell^2} \\ &= \left\langle \left( \prod_{j=1}^N P M_{u_j \circ \varphi_z} \right) (k_0 e_k), e_i \right\rangle_{\ell^2}. \end{aligned}$$

By the boundedness of the Bergman projection and the finiteness of the symbols, we deduce that  $T$  satisfies (4.1). The same argument shows that  $T$  satisfies (4.2).

Now, let  $T$  be a general operator in  $\mathcal{T}_{L_{\text{fin}}^\infty}$ . Note that if we prove that (a) holds, then it follows easily that (b) also holds. In the proof of Theorem 4.4, we proved that

$$\sup_{\|f\|_{\mathcal{B}(\Omega)} \leq 1} \limsup_{\mathfrak{d}(z,0) \rightarrow \infty} \|T^z f\|_{\mathcal{B}(\Omega)} \lesssim \|T\|_e.$$

So, we only need to prove the other inequality. Since  $T \in \mathcal{T}_{L_{\text{fin}}^\infty}$ , there is an operator,  $S$ , that is a finite sum of finite products of Toeplitz operators with  $L_{\mathcal{L}(\ell^2)}^\infty$  symbols and such that  $\|T - S\|_e \leq \|T - S\|_{\mathcal{L}(\mathcal{B}(\Omega))} < \epsilon$ . Using the fact that (a) is true for operators of this form, there holds:

$$\|T\|_e \leq \|T - S\|_e + \|S\|_e \leq \epsilon + \sup_{\|f\|_{\mathcal{B}(\Omega)} \leq 1} \limsup_{\mathfrak{d}(z,0) \rightarrow \infty} \|S^z f\|_{\mathcal{B}(\Omega)}.$$

If  $\|f\|_{\mathcal{B}(\Omega)} \leq 1$ , there holds:

$$\|S^z f\|_{\mathcal{B}(\Omega)} \leq \|(T - S)^z f\|_{\mathcal{B}(\Omega)} + \|T^z f\|_{\mathcal{B}(\Omega)} \lesssim \epsilon + \|T^z f\|_{\mathcal{B}(\Omega)}.$$

Combining the last two inequalities we obtain the desired inequality for  $T$ .  $\square$

Our next goal is to show that the previous corollary holds even with a weaker assumption. Let BUCO denote the algebra of operator-valued functions  $u : (\Omega, \mathfrak{d}) \rightarrow (\mathcal{L}(\ell^2), \|\cdot\|_{\mathcal{L}(\ell^2)})$  that are in  $L_{\text{fin}}^\infty$  and uniformly continuous. This is equivalent to requiring that  $u$  be  $d$ -finite and requiring that  $\langle u e_k, e_i \rangle_{\ell^2} \in \text{BUC}(\Omega, \mathfrak{d})$  for  $i, k = 1, \dots, d$ , where  $\text{BUC}(\Omega, \mathfrak{d})$  is the algebra of bounded uniformly continuous functions on  $\Omega$ . Let  $\mathcal{T}_{\text{BUCO}}$  denote the algebra generated by the Toeplitz operators with symbols from BUCO.

In this section we show that if  $T \in \mathcal{T}_{\text{BUCO}}$  and  $\langle T k_z e_k, k_z e_i \rangle_{\mathcal{B}(\Omega)} \rightarrow 0$  as  $\mathfrak{d}(z, 0) \rightarrow \infty$ , for every  $i, k \in \mathbb{N}$  then  $T$  is compact. Recall that for a given operator  $T$  the Berezin transform of  $T$  is an operator-valued function on  $\Omega$  given by the formula

$$\left\langle \tilde{T}(z) e_k, e_i \right\rangle_{\ell^2} := \langle T k_z e_k, k_z e_i \rangle_{\mathcal{B}(\Omega)}.$$

**Theorem 4.6.** *Let  $T \in \mathcal{T}_{\text{BUCO}}$ . Then*

$$\limsup_{\mathfrak{d}(z,0) \rightarrow \infty} \left\langle \tilde{T}(z) e_k, e_i \right\rangle_{\ell^2} = 0$$

*if and only if*

$$\limsup_{\mathfrak{d}(z,0) \rightarrow \infty} \|T^z f\|_{\mathcal{B}(\Omega)} = 0$$

for every  $f \in \mathcal{B}(\Omega)$ . In particular, if the Berezin transform  $\tilde{T}(z)$  “vanishes at the boundary of  $\Omega$ ” then the operator  $T$  must be compact.

For the remainder of this section, SOT will denote the strong operator topology in  $\mathcal{L}(\mathcal{B}(\Omega))$  and WOT will denote the weak operator topology in  $\mathcal{L}(\mathcal{B}(\Omega))$ .

The key to proving Theorem 4.6 will be the following two lemmas.

**Lemma 4.7.** *The Berezin transform is one to one. That is, if  $\tilde{T} = 0$ , then  $T = 0$ .*

*Proof.* Let  $T \in \mathcal{L}(\mathcal{B}(\Omega))$  and suppose that  $\tilde{T} = 0$ . Then there holds:

$$0 = \langle T(k_z e_k), k_z e_i \rangle_{\mathcal{B}(\Omega)} = \frac{1}{K(z, z)} \langle T(K_z e_k), K_z e_i \rangle_{\mathcal{B}(\Omega)}$$

for all  $z \in \Omega$  and for all  $i, k \in \mathbb{N}$ . In particular, there holds:

$$\frac{1}{K(z, z)} \langle T(K_z e_k), K_z e_i \rangle_{\mathcal{B}(\Omega)} \equiv 0.$$

Consider the function

$$F(z, w) = \langle T(K_w e_k), K_z e_i \rangle_{\mathcal{B}(\Omega)}.$$

This function is analytic in  $z$ , conjugate analytic in  $w$  and  $F(z, z) = 0$  for all  $z \in \Omega$ . By a standard result for several complex variables (see for instance [16, Exercise 3 p. 365]) this implies that  $F$  is identically 0. Using the reproducing property, we conclude that

$$F(z, w) = \langle T(K_w e_k)(z), e_i \rangle_{\ell^2} \equiv 0,$$

and hence

$$T(K_w e_k)(z) \equiv 0,$$

for every  $w \in \Omega$  and  $k \in \mathbb{N}$ . Since the products  $K_w e_k$  span  $\mathcal{B}(\Omega)$ , we conclude that  $T \equiv 0$  and the desired result follows.  $\square$

**Lemma 4.8.** *Let  $u \in \text{BUCO}$ . For any sequence  $\{z_n\}_{n=1}^\infty$  in  $\Omega$ , the sequence of Toeplitz operators  $T_{u \circ \varphi_{z_n}}$  has a SOT convergent subnet.*

*Proof.* Since  $u \in \text{BUCO}$ , it is finite and  $\langle T e_k, e_j \rangle_{\ell^2} \in \text{BUC}(\Omega, \mathfrak{d})$ . The result therefore follows easily from the corresponding scalar-valued case, [21, Lemma 4.7] by taking limits “entry-wise”.  $\square$

*Proof of Theorem 4.6.* Suppose that  $\limsup_{\mathfrak{d}(z,0) \rightarrow \infty} \|T^z f\|_{\mathcal{B}(\Omega)} = 0$  for every  $f \in \mathcal{B}(\Omega)$ . Take  $f \equiv e_k$  then  $\|T^z e_k\|_{\mathcal{B}(\Omega)}^2 \simeq \|T k_z e_k\|_{\mathcal{B}(\Omega)}^2$ . Then there holds that:

$$\left| \left\langle \tilde{T}(z) e_k, e_i \right\rangle_{\ell^2} \right| = \left| \langle T k_z e_k, k_z e_i \rangle_{\mathcal{B}(\Omega)} \right| \leq \|T k_z e_k\|_{\mathcal{B}(\Omega)}.$$

Therefore,  $\limsup_{\mathfrak{d}(z,0) \rightarrow \infty} \left\langle \tilde{T}(z) e_k, e_i \right\rangle_{\ell^2} = 0$  for all  $i, k \in \mathbb{N}$ .

In the other direction, suppose that  $\lim_{\mathfrak{d}(z,0) \rightarrow \infty} \left| \left\langle \tilde{T}(z) e_k, e_i \right\rangle_{\ell^2} \right| = 0$  for every  $i, k \in \mathbb{N}$  but  $\limsup_{\mathfrak{d}(z,0) \rightarrow \infty} \|T^z f\|_{\mathcal{B}(\Omega)} > 0$  for some  $f \in \mathcal{B}(\Omega)$ . In this case there exists a sequence  $\{z_n\}_{n=1}^\infty$  with  $\mathfrak{d}(z_n, 0) \rightarrow \infty$  such that  $\|T^{z_n} f\|_{\mathcal{B}(\Omega)} \geq c > 0$ . We will show that  $T^{z_n}$  has a subnet that converges to the zero operator in SOT. This, of course, will be a contradiction. Observe first that  $T^{z_n}$  has a subnet which converges in the WOT. Call this operator  $S$ . Slightly abusing notation, we continue to denote the subnet by  $\{z_n\}_{n=1}^\infty$ . Then  $\langle T^{z_n} k_z e_k, k_z e_i \rangle_{\mathcal{B}(\Omega)} \rightarrow$

$\langle Se_k, e_i \rangle_{\ell^2}$  for every  $i, k \in \mathbb{N}$ . Thus, the entries of  $\tilde{T}$  converge pointwise to the entries of  $\tilde{S}$ . More precisely, for every  $z \in \Omega$  and for every  $k, i \in \mathbb{N}$ , there holds  $\left\langle \tilde{T}^{z_n}(z)e_k, e_i \right\rangle_{\mathcal{B}(\Omega)} \rightarrow \left\langle \tilde{S}(z)e_k, e_i \right\rangle_{\ell^2}$ . The assumption  $\lim_{\mathfrak{d}(z,0) \rightarrow \infty} \left| \left\langle \tilde{T}(z)e_k, e_i \right\rangle_{\ell^2} \right| = 0$  implies that  $\left\langle \tilde{T}^{z_n}e_k, e_i \right\rangle_{\ell^2} \rightarrow 0$  pointwise for every  $i, k \in \mathbb{N}$  as well and hence  $\tilde{S} \equiv 0$ . Therefore  $S$  is the zero operator and consequently  $T^{z_n}$  converges to zero in the WOT.

Next, we use the fact that  $T$  is in  $\mathcal{T}_{\text{BUCO}}$  to show that there exists a subnet of  $T^{z_n}$  which converges in SOT. Let  $\epsilon > 0$ , then there exists an operator  $A$  which is a finite sum of finite products of Toeplitz operators with symbols in BUCO such that  $\|T - A\|_{\mathcal{L}(\mathcal{B}(\Omega))} < \epsilon$ . We first show that  $A^{z_n}$  must have a convergent subnet in SOT. By linearity we can consider only the case when  $A = T_{u_1}T_{u_2} \cdots T_{u_k}$  is a finite product of Toeplitz operators. As noticed before

$$A^{z_n} = T_{u_1 \circ \varphi_{z_n}} T_{u_2 \circ \varphi_{z_n}} \cdots T_{u_k \circ \varphi_{z_n}}.$$

Now, since a product of SOT convergent nets is SOT convergent, it is enough to treat the case when  $A = T_u$  is a single Toeplitz operator. But, the single Toeplitz operator case follows directly from Lemma 4.8.

Denote by  $B$  the SOT limit of this subnet  $A^{z_{n_k}}$ . If  $f \in \mathcal{B}(\Omega)$  is of norm at most 1, there holds:

$$\begin{aligned} \|Bf\|_{\mathcal{B}(\Omega)}^2 &= \langle Bf, Bf \rangle_{\mathcal{B}(\Omega)} \\ &\leq \left| \langle Bf - A^{z_{n_k}}f, Bf \rangle_{\mathcal{B}(\Omega)} \right| + \left| \langle T^{z_{n_k}}f - A^{z_{n_k}}f, Bf \rangle_{\mathcal{B}(\Omega)} \right| + \left| \langle T^{z_{n_k}}f, Bf \rangle_{\mathcal{B}(\Omega)} \right|. \end{aligned}$$

Using the fact that  $B$  is the SOT limit of bounded operators, we deduce that  $B$  is bounded. By the weak convergence of  $T^{z_n}$ , by taking  $n_k$  “large” enough, the outer terms above can be made less than  $\epsilon$ . By assumption, the middle term is less than  $\epsilon$ . We deduce that  $\|B\|_{\mathcal{L}(\mathcal{B}(\Omega))} \lesssim \epsilon$ . Now, for every  $f \in \mathcal{B}(\Omega)$  of norm no greater than 1 there holds

$$\|T^{z_{n_k}}f\|_{\mathcal{B}(\Omega)} \leq \|A^{z_{n_k}}f\|_{\mathcal{B}(\Omega)} + \|(T^{z_{n_k}} - A^{z_{n_k}})f\|_{\mathcal{B}(\Omega)}.$$

Therefore,  $\limsup \|T^{z_{n_k}}f\|_{\mathcal{B}(\Omega)} \lesssim \|Bf\|_{\mathcal{B}(\Omega)} + \epsilon \leq 2\epsilon$ . Finally, the fact that  $\epsilon > 0$  was arbitrary implies that  $\lim \|T^{z_{n_k}}f\|_{\mathcal{B}(\Omega)} = 0$  for all  $f$ . Consequently, we found a subnet  $T^{z_{n_k}}$  which converges to the zero operator in SOT. We are done.  $\square$

## 5. DENSITY OF $\mathcal{T}_{L_{\text{fin}}^\infty}$

In this section, we will prove that for a restricted class of Bergman-type function spaces, that an operator is compact if and only if it is in the Toeplitz algebra and its Berezin transform vanishes on  $\partial\Omega$ . See, for example, [4, 20–22, 25, 31] for similar results for the scalar-valued Bergman-type spaces.

First, let  $\mu$  be a measure on  $\Omega$ . We define the Toeplitz operator on  $\mathcal{B}(\Omega, \mathbb{C})$  with symbol  $\mu$  by:

$$(T_\mu f)(z) = \int_\Omega \langle K_w, K_z \rangle_{\mathcal{B}(\Omega, \mathbb{C})} f(w) d\mu(w).$$

Recall that a positive measure  $\mu$  on  $\Omega$  is said to be Carleson with respect to  $\sigma$  if there is a  $C$  such that for every  $f \in \mathcal{B}(\Omega, \mathbb{C})$  there holds:

$$\int_{\Omega} |f|^2 d\mu \leq C \int_{\Omega} |f|^2 d\sigma.$$

Clearly, if  $a$  is a bounded function on  $\Omega$ , then  $ad\sigma$  is Carleson with respect to  $\sigma$ .

Next, let  $\mu$  be a countably additive matrix-valued function from the Borel sets of  $\Omega$  to  $\mathcal{L}(\ell^2)$  such that  $\mu(\emptyset) = 0$ . Then we say that  $\mu$  is a matrix-valued measure. The entries of  $\mu$ , which are given by  $\langle \mu e_k, e_j \rangle_{\ell^2}$ , are all measures on  $\Omega$ . We can define a Toeplitz operator  $T_{\mu}$  on  $\mathcal{B}(\Omega)$  by the formula:

$$(T_{\mu}f)(z) = \int_{\Omega} \langle K_w, K_z \rangle_{\mathcal{B}(\Omega, \mathbb{C})} d\mu(w) f(w).$$

For this section, we define a more restrictive  $\ell^2$ -valued Bergman-type space. We add the additional assumption:

A.8 If  $\mu$  is a scalar-valued measure on  $\Omega$  whose total variation is Carleson with respect to  $\sigma$ , then  $T_{\mu} \in \mathcal{T}_{\text{BUC}}$ , where  $\mathcal{T}_{\text{BUC}}$  is the algebra of operators on  $\mathcal{B}(\Omega, \mathbb{C})$  generated by Toeplitz operators with symbols that are bounded and uniformly continuous on  $\Omega$ .

We will call such spaces  $\mathcal{B}_{\mathcal{A}}(\Omega, \mathbb{C})$  and we will call their  $\ell^2$ -valued extensions  $\mathcal{B}_{\mathcal{A}}(\Omega)$ . (The  $\mathcal{A}$  is for “approximation”.) This is not a trivial assumption and it is (at this point) not known whether this holds for all Bergman-type spaces (see [21]). It does hold in the standard Bergman spaces on the ball and polydisc and also on the Fock space see [4, 20, 22, 31]. Thus, the following theorem can be viewed as an extension of the main theorems in [4, 20, 22, 31] to the  $\ell^2$ -valued setting.

We will prove the following theorem:

**Theorem 5.1.** *Let  $T \in \mathcal{L}(\mathcal{B}_{\mathcal{A}}(\Omega))$ . Then  $T$  is compact if and only if  $\limsup_{d(z,0) \rightarrow \infty} \tilde{T}(z) = 0$  and  $T \in \mathcal{T}_{L_{\text{fin}}^{\infty}}$ .*

We will first prove the following lemma. The proof of the following lemma uses Assumption A.8.

**Lemma 5.2.** *On  $\mathcal{B}_{\mathcal{A}}(\Omega)$ ,*

$$\mathcal{T}_{\text{BUCO}} = \mathcal{T}_{L_{\text{fin}}^{\infty}}.$$

*Proof.* It is clear that  $\mathcal{T}_{\text{BUCO}} \subset \mathcal{T}_{L_{\text{fin}}^{\infty}}$ . Now, let  $u \in L_{\text{fin}}^{\infty}$ . Since  $\langle ue_k, e_j \rangle_{\ell^2}$  is bounded, for every  $k, j \in \mathbb{N}$  it follows that  $|\langle ue_k, e_j \rangle_{\ell^2}| d\sigma$  is Carleson with respect to  $\sigma$  for every  $k, j \in \mathbb{N}$ . So then the Toeplitz operator on  $\mathcal{B}_{\mathcal{A}}(\Omega, \mathbb{C})$  with symbol  $|\langle ue_k, e_j \rangle_{\ell^2}| d\sigma$  is in  $\mathcal{T}_{\text{BUC}}$ . Since  $u$  is a finite symbol, it easily follows that  $T_u \in \mathcal{T}_{\text{BUCO}}$ . Thus  $\mathcal{T}_{L_{\text{fin}}^{\infty}} \subset \mathcal{T}_{\text{BUCO}}$ . This completes the proof.  $\square$

*Proof of Theorem 5.1.* If  $T \in \mathcal{T}_{L_{\text{fin}}^{\infty}}$  then the previous lemma shows that  $T \in \mathcal{T}_{\text{BUCO}}$ . So if  $\limsup_{d(z,0) \rightarrow \infty} \tilde{T}(z) = 0$  then by Theorem 4.6,  $T$  is compact. On the other hand, if  $T$  is compact, then by Lemma 4.1 there holds that  $\limsup_{d(z,0) \rightarrow \infty} \tilde{T}(z) = 0$ . So, we only need to show that  $T$  is in  $\mathcal{T}_{L_{\text{fin}}^{\infty}}$ . Since  $T$  is compact, it suffices to show that each rank 1 operator is in  $\mathcal{T}_{L_{\text{fin}}^{\infty}}$ . The rank 1 operators are given by the formula:

$$(f \otimes g)(h) = \langle h, g \rangle_{\mathcal{B}(\Omega)} f.$$

So, we need to show that  $f \otimes g \in \mathcal{T}_{L_{\text{fin}}^\infty}$ . Let  $p_f$  be a polynomial such that  $\|f - p_f\|_{\mathcal{B}_A(\Omega)} < \frac{\epsilon}{\|g\|_{\mathcal{B}_A(\Omega)}}$  and  $p_g$  a polynomial such that  $\|g - p_g\|_{\mathcal{B}_A(\Omega)} < \frac{\epsilon}{\|p_f\|_{\mathcal{B}_A(\Omega)}}$ . Then for  $h \in \mathcal{B}_A(\Omega)$  there holds:

$$\begin{aligned} \|(f \otimes g)h - (p_f \otimes p_g)h\|_{\mathcal{B}_A(\Omega)} &= \left\| \langle h, g \rangle_{\mathcal{B}_A(\Omega)} f - \langle g, p_g \rangle_{\mathcal{B}_A(\Omega)} p_f \right\|_{\mathcal{B}_A(\Omega)} \\ &\leq \left\| \langle h, g \rangle_{\mathcal{B}_A(\Omega)} (f - p_f) \right\|_{\mathcal{B}_A(\Omega)} + \left\| \langle h, g - p_g \rangle_{\mathcal{B}_A(\Omega)} p_f \right\|_{\mathcal{B}_A(\Omega)} \\ &\leq 2\epsilon \|h\|_{\mathcal{B}_A(\Omega)}. \end{aligned}$$

Therefore, if we can show that  $p_f \otimes p_g \in \mathcal{T}_{L_{\text{fin}}^\infty}$ , then we will be finished. For the following computation, we use the following notational convinienicies. Let  $E_{i,j}$  be the matrix such that  $\langle E_{i,j} e_k, e_l \rangle = 1$  when  $k = j$  and  $l = i$  and zero otherwise. That is,  $E_{i,j}$  is the matrix with a 1 in the  $(i, j)$  position and zeros everywhere else. We abuse notation and write  $f$  in place of  $p_f$  and  $g$  in place of  $p_g$ , keeping in mind that this means that  $f$  and  $g$  are both now polynomials. Lastly, we will abuse notation again and  $P$  will also denote the projection from  $L^2(\Omega, \mathbb{C}; d\sigma)$  onto  $\mathcal{B}_A(\Omega, \mathbb{C})$ . Observe that if  $f \in \mathcal{B}_A(\Omega)$  then:

$$Pf = \sum_{i=1}^{\infty} (Pf_i) e_i,$$

where on the left hand side,  $P$  is the projection on  $L^2(\Omega, \ell^2; d\sigma)$  and on the right hand side  $P$  is the projection on  $L^2(\Omega, \mathbb{C}; d\sigma)$ . Observe that since  $K_0(z) = \|K_0\|_{\mathcal{B}_A(\Omega, \mathbb{C})} k_0(z)$  and since  $k_0 \equiv 1$  on  $\Omega$ , there holds that  $K_0 \equiv \|K_0\|_{\mathcal{B}_A(\Omega, \mathbb{C})}$ .

Using these facts, we compute:

$$\begin{aligned} \sum_{i=1}^{\infty} \sum_{k=1}^{\infty} (T_{f_i E_{i,i}} T_{\delta_0 E_{i,i}} T_{\overline{g_k} E_{i,i}} T_{E_{i,k}} h)(z) &= \sum_{i=1}^{\infty} \sum_{k=1}^{\infty} (f_i T_{\delta_0 E_{i,i}} T_{\overline{g_k} E_{i,i}} T_{E_{i,k}} h)(z) \\ &= \sum_{i=1}^{\infty} \sum_{k=1}^{\infty} f_i(z) (T_{\overline{g_k} E_{i,i}} T_{E_{i,k}} h)(0) \\ &= \sum_{i=1}^{\infty} \sum_{k=1}^{\infty} f_i(z) P(\overline{g_k} h_k e_i)(0) \\ &= \sum_{i=1}^{\infty} f_i(z) \sum_{k=1}^{\infty} \int_{\Omega} h_k(w) \overline{g_k(w)} K_0(w) d\sigma e_i \\ &= \|K_0\|_{\mathcal{B}_A(\Omega, \mathbb{C})} \sum_{i=1}^{\infty} f_i(z) \int_{\Omega} \sum_{k=1}^{\infty} h_k(w) \overline{g_k(w)} d\sigma e_i \\ &= \|K_0\|_{\mathcal{B}_A(\Omega, \mathbb{C})} \langle h, g \rangle f(z) = \|K_0\|_{\mathcal{B}_A(\Omega, \mathbb{C})} (f \otimes g)h(z). \end{aligned}$$

We therefore conclude that:

$$\sum_{i=1}^{\infty} \sum_{k=1}^{\infty} \left( T_{f_i E_{i,i}} T_{\|K_0\|_{\mathcal{B}_A(\Omega, \mathbb{C})}^{-1} \delta_0 E_{i,i}} T_{\overline{g_k} E_{i,i}} T_{E_{i,k}} h \right)(z) = (f \otimes g)h(z).$$

Since pointwise evaluation is a bounded linear functional, we conclude that  $\|K_0\|_{\mathcal{B}_A(\Omega, \mathbb{C})}^{-1} \delta_0$  is a Carleson measure for  $\mathcal{B}_A(\Omega, \mathbb{C})$  with respect to  $\sigma$ . Thus,  $T_{\|K_0\|_{\mathcal{B}_A(\Omega, \mathbb{C})}^{-1} \delta_0 E_{i,i}} \in \mathcal{T}_{L_{\text{fin}}^\infty}$  for every

$i \in \mathbb{N}$ . Furthermore, each of the operators  $T_{f_i E_{i,i}}$ ,  $T_{\overline{g_k} E_{i,i}}$ , and  $T_{E_{i,k}}$  are Toeplitz operators with symbols in BUCO and so each one is in  $\mathcal{T}_{L_{\text{fin}}^\infty}$ . Since  $f_i$  and  $g_i$  are finite, the sums above are finite. This implies that the operator given by the formula

$$\sum_{i=1}^{\infty} \sum_{k=1}^{\infty} \left( T_{f_i E_{i,i}} T_{\|K_0\|_{\mathcal{B}_{\mathcal{A}}(\Omega, \mathbb{C})}^{-1}}^{\delta_0 E_{i,i}} T_{\overline{g_k} E_{i,i}} T_{E_{i,k}} h \right)$$

is a member of  $\mathcal{T}_{L_{\text{fin}}^\infty}$  and therefore,  $(f \otimes g)$  is in  $\mathcal{T}_{L_{\text{fin}}^\infty}$  for all polynomials,  $f$  and  $g$ . This completes the proof.  $\square$

## 6. ACKNOWLEDGEMENTS

The author would like to thank Michael Lacey for supporting him as a research assistant for the Spring semester of 2014 (NSF DMS grant #1265570) and Brett Wick for supporting him as a research assistant for the Summer semester of 2014 (NSF DMS grant #0955432) and for discussing the problem with him.

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